Supplementary Exercises for *Mechanics* by Landau & Lifshitz

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The exercises presented here are intended for use with the classic textbook, *Mechanics*, by L. Landau and E.M. Lifshitz (hereafter LL). Many of the exercises are themselves classics and found in other textbooks. The purpose of this small book is to present them in a sequence, format and notation that parallels LL. I use these exercises in my own classes and hope other instructors will find them useful for theirs. To students using LL in their studies: You're in for a treat.

The exercises are divided by chapters in LL with a few miscellaneous exercises included at the end. Most are intended for homework with a few shorter exercises suitable for self-review or in-class exams. Some exercises require simple programming commensurate with the computer skills expected of advanced students in physics.

Comments, suggestions, and especially, new exercises are always welcome; please contact me at `algarcia@algarcia.org`.

1 The Equations of Motion

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1. Consider two disks of radius R, the first rolls on a horizontal surface and the second rolls on top of the first (see Fig. 1). We may uniquely define the positions by specifying the Cartesian co-ordinates of the centers of the disks, (x_1, y_1) and (x_2, y_2), respectively. However, since the disks roll without slipping, given the constraints that the first disk remains in contact with the ground and the second disk remains in contact with the first, we are left with only two degrees of freedom (for simplicity we'll only consider the case where the second disk is above the ground).

Suppose we take as our generalized coordinates, q_1 and q_2, the angles of rotation for each disk (see Fig. 1). Take the Cartesian origin at the base of the first disk before rotation so if $q_1 = q_2 = 0$ then $x_1 = 0$, $y_1 = R$, $x_2 = 0$, and $y_2 = 3R$.

(a) Express the Cartesian co-ordinates of the first disk in terms of q_1, that is, find $x_1(q_1)$ and $y_1(q_1)$.

(b) Find $x_2(q_1, q_2)$ and $y_2(q_1, q_2)$. Check that your expressions give $x_2 = x_1$ and $y_2 = 3R$ when $q_1 + q_2 = 0$; also check that they give $x_2 = x_1 + 2R$ and $y_2 = R$ when $q_1 + q_2 = 90°$.]

(c) Repeat part (b) for the case where the disks have different radii, R_1 and R_2.

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2. Figure 2 shows a rectangular block (mass m_2, width w_2, height h_2) that slides freely on an inclined plane (mass m_1, width w_1, height h_1, angle $\theta = \text{atan}(h_1/w_1)$). The plane itself slides freely on a horizontal surface. For simplicity, only consider the motion when the block is on the plane (i.e., before it reaches the bottom of the incline).

The position of the inclined plane may be given by the rectangular coordinates of its lower, left corner, x_1 and y_1 (see the illustration below). Similarly, call x_2 and y_2 the coordinates of the lower, left corner of the block.

(a) Define the following generalized co-ordinates: q_1 is the horizontal displacement of the inclined plane; q_2 is the distance that the right edge of the block has traveled down the plane. Express q_1 and q_2 in terms of x_1, y_1, x_2, and y_2.

(b) Express x_1, y_1, x_2, and y_2 in terms of q_1 and q_2.

(c) The total kinetic energy for the system is

$$T = \tfrac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

Find $T(q_1, q_2, \dot{q}_1, \dot{q}_2)$, that is, express the kinetic energy as a function of the generalized coordinates.

(d) Consider the rectangular coordinates, x_1', y_1', x_2', and y_2', defined by the location of the upper, right corner of each object. Repeat parts (a)-(c) for these coordinates.

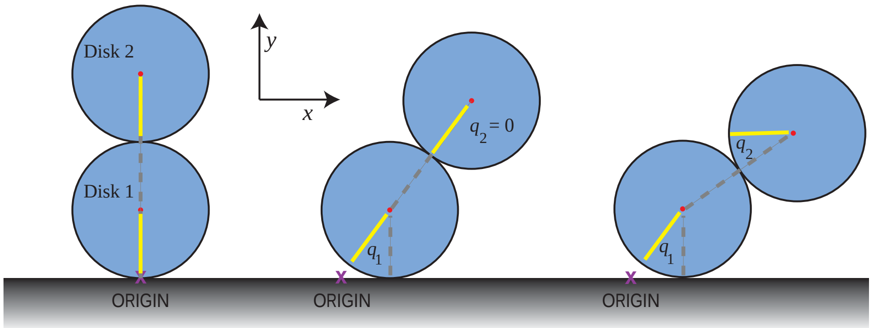

Figure 1: Two disks of radius R, the first rolls on a horizontal surface and the second rolls on top of the first disk.

3. Consider the block and inclined plane system from the previous exercise (see Fig. 2). There is no friction but a constant downward gravitational force (i.e., acceleration g) acts on the two objects.

(a) For this system, find the Lagrangian, $L(q_1, q_2, \dot{q}_1, \dot{q}_2)$, as $L = T - U$ where U is the gravitational potential energy and T is the kinetic energy (see previous exercise).

(b) Using Lagrange's equation find the first equation of motion and express it in the form $\ddot{q}_1 = f_1(\ddot{q}_2, m_1, m_2, \theta)$.

(c) Using Lagrange's equation find the second equation of motion and express it in the form $\ddot{q}_2 = f_2(g, m_1, m_2, \theta)$.

(d) Graph $\ddot{q}_2/g$ versus θ for the following cases: $m_1 \gg m_2$; $m_1 = m_2$; and $m_1 \ll m_2$. Use a logarithmic scale on the vertical axis and a linear scale on the horizontal axis (i.e., use a semi-log scale).

(e) Describe, in general terms, how the objects move in the limits $m_1 \gg m_2$ and $m_1 \ll m_2$.

4. (a) From (5.10) in LL, the Lagrangian for a particle moving under a constant force is,

$$L(\mathbf{r}, \mathbf{v}) = \tfrac{1}{2}mv^2 + \mathbf{F} \cdot \mathbf{r}$$

Use this to formulate the Lagrangian for the simple pendulum in the $x - y$ plane following the notation in Problem 1 on pg. 11 (take $m_1 = m$, $\ell_1 = \ell$, $m_2 = 0$, $\ell_2 = 0$).

(b) Formulate the Lagrangian in Problem 1 on pg. 11, filling in any steps that LL skip.

5. (a) Consider a system with the following Lagrangian (see Problem 2, page 11 of LL)

$$L = \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}m_2(\dot{x}^2 + 2\dot{x}\dot{\phi}\ell\cos\phi + \dot{\phi}^2\ell^2) + m_2 g\ell\cos\phi$$

where $q_1 = x$ and $q_2 = \phi$ are the generalized co-ordinates. Using Lagrange's equations, find the equations of motion and show that if $\ddot{x} = 0$ then one of the equations of motion reduces to the equation of motion of a simple pendulum and the other equation of motion reduces to the result that the horizontal acceleration of particle 2 is zero.

(b) Consider a system with the following Lagrangian (see Problem 4, page 12 of LL)

$$L = m_1 a^2(\dot{\theta}^2 + \Omega^2\sin^2\theta) + 2m_2 a^2\dot{\theta}^2\sin^2\theta + 2(m_1 + m_2)ga\cos\theta$$

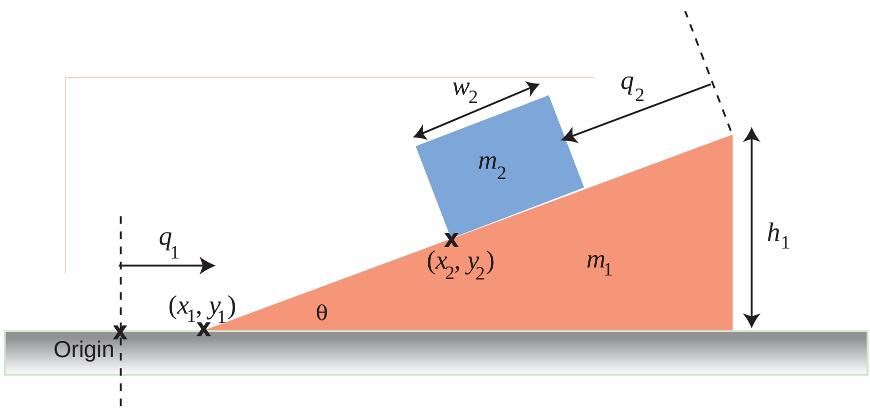

Figure 2: Rectangular block (mass m_2, width w_2, height h_2) slides freely on an inclined plane (mass m_1, width w_1, height h_1, angle $\theta = \text{atan}(h_1/w_1)$). The plane itself slides freely on a horizontal surface.

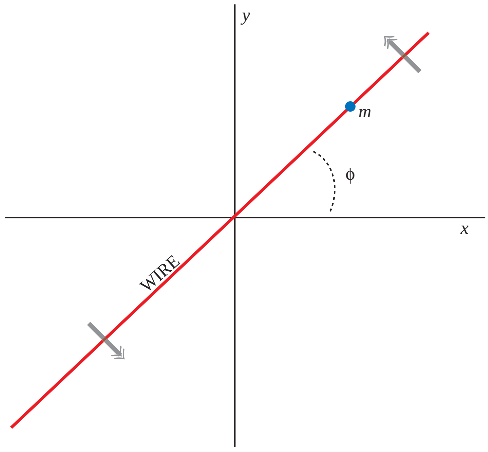

Figure 3: Bead, mass m, sliding freely on a straight horizontal wire rotating with angular velocity Ω.

where $q_1 = \theta$ is the generalized co-ordinate. Using Lagrange's equations, find the equation of motion and show that if $\ddot{\theta} = \dot{\theta} = 0$ then

$$\theta = \mathrm{acos}\frac{(m_1 + m_2)g}{m_1 a \Omega^2}$$

which is the steady-state angle (i.e., the angle if the motion is steady).

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6. Consider a bead, mass m, sliding freely on a straight wire; the wire rotates about the origin in the xy plane with a constant angular velocity Ω.

(a) Find the Lagrangian, $L(r, \dot{r})$.
(b) From the Lagrangian, find the equation of motion.
(c) Solve the equation of motion to obtain $r(t)$.
(d) From part (c), find the initial velocity for which $r(t) \to 0$ as $t \to \infty$.
(e) Show that for $t \ll |\Omega|^{-1}$ the solution from part (c) is

$$r(t) \approx r_0 + v_0 t + \frac{1}{2} a_0 \Omega^2 t^2,$$

where $r_0 = r(0)$ and $v_0 = \dot{r}(0)$ are the initial position and velocity. That is, the particle initially moves with an approximately constant acceleration of a_0; obtain an explicit expression for a_0.

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7. Consider a particle (mass m) constrained to move along a straight line (like a bead on a frictionless wire); see Fig. 4. The line is at a fixed angle α from the z-axis and rotates with constant angular velocity, Ω, so $\dot{\phi} = \Omega$. A constant external field acts on the particle, exerting a downward force of magnitude mg.

(a) Find the Lagrangian $L(z, \dot{z}) = T - U$ where T is the kinetic energy of the particle and $U = mgz$ is the potential energy.

(b) From the Lagrangian obtain (but do not solve) the equation of motion.

(c) From the equation of motion show that a particle initially at rest and located at $z = z_0$ where $z_0 = g/(\Omega^2 \tan^2 \alpha)$ will remain at z_0 and that a particle initially at rest and located above (or below) the point z_0 will move continuously upward (or downward).

(d) Solve the equation of motion and obtain an explicit expression for $z(t)$, fixing the constants in the solution of the ODE by using the initial conditions, $z(0)$ and $\dot{z}(0)$.

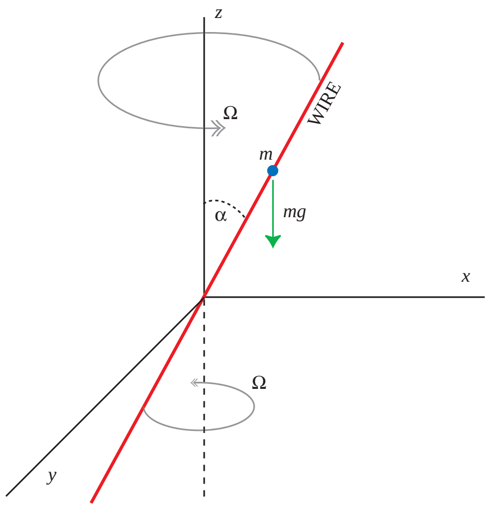

Figure 4: Bead (mass m) constrained to move on a wire at fixed angle α that rotates with angular velocity, Ω; downward gravity force is mg.

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8. Consider a variant of the system in Problem 4 (pg. 12 in LL) for which two point masses, m_1, are attached to an axle by massless rods of *different* lengths. The upper rods (length a_1) are attached to the axle at a fixed hinge point so the angle θ is allowed to vary. The lower rods (length a_2) are attached to a point mass m_2, which freely slides up and down along the axis. The axle and the masses m_1 rotate about the vertical axis with constant angular velocity $\dot{\phi} = \Omega$.

(a) Taking the top hinge as the origin and the z axis pointing up the axis, express the x, y, and z coordinates of the masses in terms of the spherical co-ordinate angles, θ and ϕ.

(b) Find the Lagrangian $L(\theta, \phi, \dot{\theta}, \dot{\phi})$ taking gravitational acceleration as $-g\hat{z}$.

(c) Show that your result from part (b) agrees with the Lagrangian given in your textbook for the case where $a_1 = a_2 = a$.

(d) Find the equation of motion for $\theta(t)$ when $a_1 = a_2 = a$.

(e) Using your result from part (d), find the steady state angle θ_0, that is, if $\ddot{\theta} = \dot{\theta} = 0$ then find $\theta(t) = \theta_0$. From your result compute the steady state angle for $m_1 = m_2$ when $a\Omega^2 = g$ and when $a\Omega^2 = 4g$.

2 Conservation Laws

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1. Consider a system with the following Lagrangian, expressed in Cartesian co-ordinates,

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \tfrac{1}{2}k(x^2 + y^2)$$

where m and k are constants.

(a) Is the energy, E, conserved? Justify your answer.

(b) Is the x-component of linear momentum, $p_x = \mathbf{P}\cdot\hat{\mathbf{x}}$, conserved? Is the z-component of linear momentum, $p_z = \mathbf{P}\cdot\hat{\mathbf{z}}$, conserved?

(c) Is the x-component of angular momentum, $M_x = \mathbf{M}\cdot\hat{\mathbf{x}}$, conserved? Is the z-component of angular momentum, $M_z = \mathbf{M}\cdot\hat{\mathbf{z}}$, conserved?

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2. Consider a free particle constrained to move on the surface of a cylinder of radius R whose central axis is along the z-axis. There is no gravitational potential.

(a) Write down the Lagrangian and, from inspection, explain why we immediately know that the energy, E, and the z component of angular momentum, M_z, are constant in time.

(b) Using the equation of motion explicitly confirm that E and M_z are constant in time.

(c) The x component of angular momentum, M_x, is *not* constant in time. Find its time derivative and express it as $\dot{M}_x = f(\phi, \dot{\phi}, z)$. For what special cases is M_x constant?

(d) Finally, consider a free particle constrained to move on the surface of a sphere of radius R whose center is at the origin. Explicitly show that the magnitude of the angular momentum is constant.

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3. Consider a system with the following Lagrangian (see Problem 4, page 12)

$$L(\theta, \dot{\theta}) = m_1 a^2(\dot{\theta}^2 + \Omega^2 \sin^2\theta) + 2m_2 a^2\dot{\theta}^2 \sin^2\theta + 2(m_1 + m_2)ga\cos\theta$$

(a) Find the energy of the system, $E(\theta, \dot{\theta})$. Note that the Lagrangian does *not* depend explicitly on time so this energy is constant (i.e., integral of the motion).

(b) Find the kinetic energy, $T(\theta, \dot{\theta})$ and potential energy $U(\theta)$.

(c) Show that $E \neq T + U$; this is not a contradiction of equation (6.2) since this mechanical system is *not* closed.

(d) Find the generalized momentum and the generalized force.

(e) For this system, under what conditions is the generalized momentum constant?

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4. Consider the following system (see Fig. 5): Two particles (masses m_1 and m_2) are connected by a string of length ℓ. The first particle moves freely on the xy plane (as on a table top) while the second moves freely along the z axis (the string passes through a hole at the origin). Gravitational acceleration, g, acts in the negative z direction.

(a) Write the Lagrangian $L(r, \theta, \dot{r}, \dot{\theta})$ where r and θ are the cylindrical co-ordinates of the first particle.

(b) Find the energy E and M_z, the angular momentum about the z axis and explain how we know that they are constant for this system. Are there any other integrals of the motion?

(c) Show that the motion may be expressed in the form $\dot{r} = f(r, E, M_z)$.

(d) Graph $|\dot{r}|$ versus r taking the values $m_1 = m_2 = 1$, $g = 1$, $M_z = 1$, for the cases $E = 1$, 2, 4, and 8.

(e) For $E = 1.0, 1.1, \ldots, 7.9, 8.0$, graph the minimum and maximum values of r. Take $m_1 = m_2 = 1$, $g = 1$, $M_z = 1$.

3 Integration of the Equations of Motion

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1. Consider a particle (mass m and energy E) moving in one-dimensional motion with the potential

$$U(x) = \begin{cases} mgx & x > 0 \\ \infty & x < 0 \end{cases}$$

where $g > 0$.

(a) Find the turning points, x_1 and x_2, of the motion.

(b) Find the period of oscillation $T(E)$.

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2. A particle (mass m, position x) moving in a potential $U(x)$ is found to have a period that depends on its energy E as

$$T = \pi\sqrt{\frac{2m\ell^2}{U_0 - E}} \qquad 0 \le E < U_0$$

where ℓ and U_0 are constants, with units of length and energy, respectively.

(a) Find the potential $U(x)$; you may assume that it is symmetric. [Hint: You will probably need to consult integral tables and maybe look up some hyperbolic trig identities.]

(b) Sketch the potential $U(x)$ versus x, indicating $U = U_0$ and $x = \pm\ell$ on your graph.

(c) Show that in the limit $E \ll U_0$, the potential is approximately quadratic; writing the potential of simple harmonic motion as $U_{\rm SHM}(x) = \frac{1}{2}kx^2$, find k in terms of U_0 and ℓ. Show that in this limit the period is that of simple harmonic motion, which is $T_{\rm SHM} = 2\pi\sqrt{m/k}$.

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3. In Problem 2, pg. 40, Landau and Lifshitz give the solution for a particle moving in a central field $U = -\alpha/r^2$, $\alpha > 0$. Specifically, for bounded motion ($E < 0$), they give the results for $r(\phi)$ as,

$$\frac{1}{r} = \sqrt{\frac{|E|}{\alpha - M^2/2m}}\cosh\left[\phi\sqrt{\frac{2m\alpha}{M^2} - 1}\right]$$

and for $t(r)$ as,

$$t = \frac{1}{|E|}\sqrt{\frac{m}{2}\left(Er^2 - \frac{M^2}{2m} + \alpha\right)}$$

<u>Part A</u>: This part requires using a computer to produce several graphs. For all the computations below, take the values $E = -1/4$, $m = 1$, $\alpha = 4$, $M^2/2m = 3$.

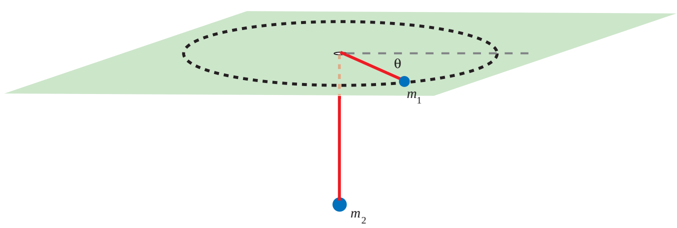

Figure 5: Two particles (masses m_1, m_2) connected by a string (length ℓ). The first moves on the xy plane; the second moves along the z axis (downward gravitational acceleration, g).

(a) Compute r for various values of ϕ (at least 20 values between $\phi = 0$ and 3π) and make an accurate graph of r versus ϕ.

(b) Graph the trajectory by graphing (r, ϕ) in polar coordinates or converting and graphing the values in Cartesian co-ordinates, (x, y).

(c) For the values of r obtained in part (a), find the corresponding values of t and graph r versus t.

(d) Using the results of parts (a) and (c), graph ϕ versus t.

If you write a computer program, include a listing of your program; if you use a simple calculator, include your tables of values. [Hint: With the parameter values above, the particle reaches the origin at time $t = \sqrt{8}$.]

Part B: Use equations (14.6) and (14.7) to derive the equations for $r(\phi)$ and $t(r)$ given above.

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4. Consider a comet in a parabolic orbit as it passes near the sun (i.e., Kepler problem with $E = 0$). Assuming that the Earth's orbit is circular with radius a.

(a) Show that the time during which the comet is within Earth's orbit (i.e., $r \leq a$) may be written as

$$t_{\text{in}} = \frac{\sqrt{2}}{3\pi} T \left(1 + \frac{2r_{\text{min}}}{a}\right) \sqrt{1 - \frac{r_{\text{min}}}{a}}$$

where r_{min} is the minimum distance from the comet to the sun (perihelion) and T is the period of Earth's orbit.

(b) Find the maximum time, in days, that a comet in a parabolic orbit can be within Earth's orbit.

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5. Consider a particle, initially at rest, in the attractive central potential $U(r) = -\alpha/r$ at a distance $r = R$ from the origin. Find the time it takes for the particle to reach the origin and express your result in terms of T, the period of a circular orbit of radius R for a particle of the same mass.

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6. Consider the three-body problem in which three particles (masses m_1, m_2, and m_3, positions $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$) interact by mutual gravitational attraction. The equation of motion for the first particle may be written as

$$\ddot{\mathbf{r}}_1 = -Gm_2 \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} - Gm_3 \frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3}$$

with similar equations of motion for the other two particles.

(a) Define the relative position vectors for the three-body problem as

$$\mathbf{s}_1 = \mathbf{r}_2 - \mathbf{r}_3; \qquad \mathbf{s}_2 = \mathbf{r}_3 - \mathbf{r}_1; \qquad \mathbf{s}_3 = \mathbf{r}_1 - \mathbf{r}_2$$

Show that the equations of motion for the relative position vectors are

$$\ddot{\mathbf{s}}_i = -\mu G \frac{\mathbf{s}_i}{|\mathbf{s}_i|^3} + m_i G \mathbf{A} \qquad (i = 1, 2, 3)$$

where $\mu = m_1 + m_2 + m_3$ and

$$\mathbf{A} = \left(\frac{\mathbf{s}_1}{|\mathbf{s}_1|^3} + \frac{\mathbf{s}_2}{|\mathbf{s}_2|^3} + \frac{\mathbf{s}_3}{|\mathbf{s}_3|^3} \right)$$

You only need to show this for $i = 1$ since the result is similar for $i = 2$ and 3.

(b) Show that when $m_1 = m_2 = m_3$ and $\mathbf{A} = 0$ the three bodies can travel in circular orbits around the center-of-mass with the relative position vectors forming an equilateral triangle. (Hint: You might get some clues from the picture below, which shows the more complicated case of $m_1 = m_2/2 = m_3/3$ and $\mathbf{A} = 0$).

4 Collisions between Particles

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1. Consider the elastic collision of two particles, as described in Section 17. Show that if $m_1 = m_2$ and $|\mathbf{p}_1 + \mathbf{p}_2| = 2|m\mathbf{v}|$ then the velocities of the particles after the collision, $\mathbf{v}'_1$ and $\mathbf{v}'_2$, are perpendicular.

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2. In section 17 Landau and Lifshitz analyze the elastic collision of two particles. Consider the case of an inelastic collision in which the total momentum is conserved, that is,

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}'_1 + m_2\mathbf{v}'_2$$

but a fraction, $a \leq 1$, of the internal kinetic energy is lost in the collision. Calling $e = \sqrt{a}$ the coefficient of restitution then

$$e^2\left(\tfrac{1}{2}m_1v_{10}^2 + \tfrac{1}{2}m_2v_{20}^2\right) = \tfrac{1}{2}m_1{v'}_{10}^2 + \tfrac{1}{2}m_2{v'}_{20}^2$$

where the subscript 0 indicates velocities in the center of mass frame of reference (the C system).

Generalize the results on pages 44 and 45 to inelastic collisions. Specifically:

(a) Reformulate equation (17.1) to include the coefficient of restitution.

(b) Reformulate equations (17.2), (17.3) and Figure 15 in LL to include the coefficient of restitution.

(c) Reformulate (17.4) and Figure 16 in LL to include the coefficient of restitution.

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3. Consider two cases of a particle (mass m_1) moving in the $+x$ direction colliding with a stationary particle (mass m_2).

(a) Take $m_2 \gg m_1$ and suppose the light particle is deflected by the collision such that it moves in the $+y$ direction. Find $\mathbf{v}'_2$, that is, the magnitude and direction of the velocity of the massive particle after collision, in terms of m_1, m_2, and v_1, the magnitude of the velocity of the light particle before collision.

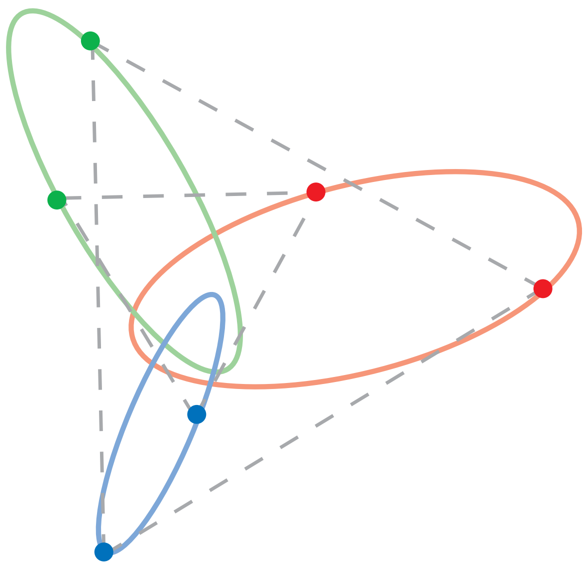

Figure 6: Three-body problem of masses m_1, m_2, and m_3 (positions $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$) interacting by mutual gravitational attraction.

(b) Take $m_2 \ll m_1$ and suppose the collision is such that the deflection angle of the massive particle is maximum. Find $\mathbf{v}_2'$, that is, the magnitude and direction of the velocity of the light particle after collision, in terms of m_1, m_2, and v_1, the magnitude of the light particle before collision.

5 Small Oscillations

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1. Consider the system in Problem 2 on page 11 in LL *but* with the point of support constrained to move with a constant acceleration a_0, that is, $x(t) = \frac{1}{2}a_0t^2$.

(a) Find the equation of motion in the small angle approximation, keeping only terms linear in ϕ.

(b) Solve the equation of motion from part (a) to obtain $\phi(t)$ taking the initial condition $\phi(0) = \dot{\phi}(0) = 0$, i.e., initially at rest.

(c) Suppose that we include a frictional force of the form $f_{\rm fr} = -\alpha\dot{\phi}$. As $t \to \infty$, $\phi(t) \to \phi_0$; find this steady state angle, ϕ_0 from the equation of motion *without* using the small angle approximation.

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2. In center-of-mass coordinates, the Lagrangian for a non-rotating diatomic molecule (reduced mass m) may be written as

$$L(r) = T(\dot{r}) - U(r) = \frac{1}{2}m\dot{r}^2 - 4\epsilon\left(\frac{\sigma^{12}}{r^{12}} - \frac{\sigma^6}{r^6}\right)$$

where the first term is the kinetic energy for the separation coordinate r and the second is the Lennard-Jones potential (a common approximation in molecular physics). Note that ϵ and σ are constants.

(a) Find the stable equilibrium distance, r_0, in terms of σ. Hint: Since the potential is minimum at r_0, it may be found by evaluating

$$\left[\frac{dU(r)}{dr}\right]_{r=r_0} = 0$$

(b) Taylor expand the potential in the form

$$U(r) = U(r_0) + \tfrac{1}{2}k(r - r_0)^2 + ...$$

to find k. Note that for such a molecule the frequency of small oscillations will be $\omega = \sqrt{k/m}$.

(c) Make an accurate graph of the Lennard-Jones potential and its small oscillation approximation, $U_{\rm so} = U(r_0) + \frac{1}{2}k(r - r_0)^2$. Take the range from $r = 0.9\sigma$ to 1.5σ; indicate the value of r_0 on your graph.

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3. Consider a pendulum (see Fig. 7 that is constructed by attaching a particle (mass m) to a string of length ℓ_0. The upper end of the string is connected to the uppermost point on a vertical disk of radius R, as shown below. The pendulum swings back and forth but the string is long enough and amplitude of the oscillations is small enough that the particle never touches the disk.

(a) Find the Lagrangian in terms of the angle θ and show that it may be written as

$$L = \frac{1}{2}m(A + B\theta)^2\dot{\theta}^2 + mg[(A + B\theta)\sin\theta + B\cos\theta]$$

and obtain explicit expressions for the constants A and B. [Hint: Careful with setting this up; easy to make errors.]

(b) The system has an equilibrium position at θ_0; obtain an explicit expression for this angle.

(c) Find the frequency of small oscillations about θ_0.

(d) Obtain an expression for the frequency, $\omega(\theta_{\rm max}, \theta_{\rm min})$, where $\theta_{\rm max}$ and $\theta_{\rm min}$ are the maximum and minimum angles of the motion, *without* the small oscillations approximation. You may leave your result in the form of a definite integral.

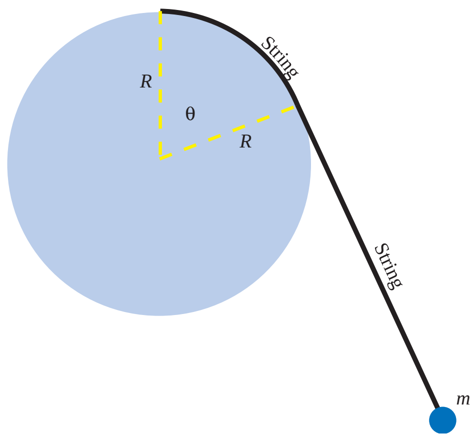

Figure 7: Pendulum consisting of a particle (mass m) on a string (length ℓ_0) connected to the top of a disk (radius R).

4. Consider a particle, mass m, that oscillates freely with frequency ω. Initially the particle is at rest ($x(0) = \dot{x}(t) = 0$) but an external force acts on it to produce forced oscillations. Specifically, the force is a square-wave pulse of the form

$$F(t) = \begin{cases} F_0 & 0 < t < T \\ -F_0 & T < t < 2T \\ 0 & \text{otherwise} \end{cases}$$

where F_0 is a constant.

(a) Find the motion $x(t)$ for $t > 2T$ (i.e., after the pulse strikes the system) taking $\omega T = \pi$.

(b) Compare the energy imparted to the particle by the above square-wave pulse to that imparted by a constant square pulse,

$$F(t) = \begin{cases} F_0 & 0 < t < 2T \\ 0 & \text{otherwise} \end{cases}$$

Again, take $\omega T = \pi$.

---

5. Consider two particles (mass m) connected to springs, as shown in Fig. 8. The spring between the particles has spring constant k_a while the other two springs have spring constant k_b; the ends of the latter are fixed to stationary supports. The system is at rest when the displacements of the particles, x_1 and x_2, are zero.

(a) Find the Lagrangian for this system.

(b) Find the frequencies of oscillation for this system.

(c) Find the eigenvectors associated with the frequencies of oscillation and from them sketch the motion associated with each of the modes of oscillation.

---

6. A homogeneous, rigid bar (length ℓ, mass m) is suspended from a ceiling by a pair of springs (spring constant k) attached to each end of the bar (see Fig. 9). The amplitude of the motion is small such that the displacements of the ends from equilibrium rest positions, y_1 and y_2, are small (i.e., $y_1, y_2 \ll \ell$); the horizontal displacement of the bar is negligible.

(a) Find the Lagrangian $L(y_1, y_2, \dot{y}_1, \dot{y}_2)$.

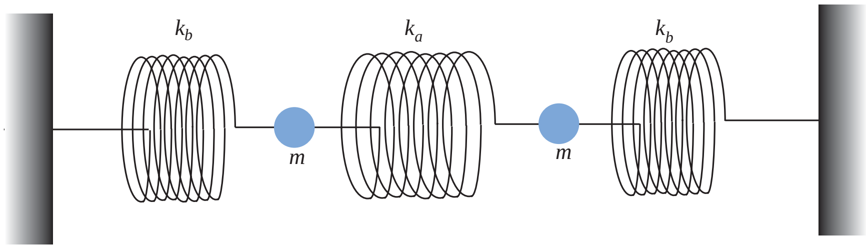

Figure 8: Springs connected to masses and stationary supports.

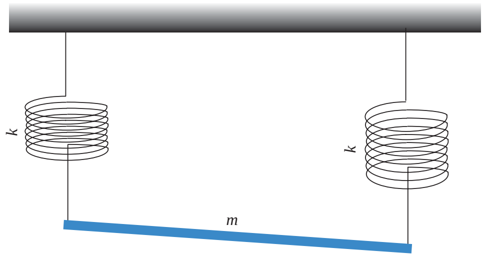

Figure 9: Homogeneous, rigid bar (length ℓ, mass m) hanging a pair of springs (stiffness k).

(b) Find the frequencies of small oscillations.

(c) A particle of mass M is attached to the center of the rod such that one frequency is half of the other. Find M in terms of m.

(d) Suppose that a constant gravitational force, with acceleration g, acts in the $-y$ direction. Do the frequencies found in parts (b) and (c) do not depend on g? Justify your answer.

---

7. Consider a simple pendulum of mass m_2 and length ℓ, with a mass m_1 at the point of support, which can move on a horizontal line lying in the plane in which m_2 moves (see Fig. 10). Landau and Lifshitz give the Lagrangian (pg. 11) as

$$L = \tfrac{1}{2}(m_1 + m_2)\dot{x}^2 + \tfrac{1}{2}m_2(\ell^2\dot{\phi}^2 + 2\ell\dot{x}\dot{\phi}\,\cos\phi) + m_2 g\ell\cos\phi$$

(a) Taking the steady state as $x_0 = 0$ and $\phi_0 = 0$, obtain explicit expressions for the coefficients m_{ik} and k_{ik} in the small oscillations approximation.

(b) Find the two eigenfrequencies for small oscillations in this system.

(c) Taking $m_1 = m_2$, find the coefficients, A_k, of the eigenvectors that correspond to the eigenfrequencies. Do not bother normalizing these eigenvectors.

(d) From the result obtained in part (c), describe the motion of the system when it oscillates in the normal mode with the higher frequency (take $m_1 = m_2$). Draw a simple diagram illustrating your description.

---

8. Consider a system of s identical particles (mass m) coupled by springs (stiffness k) with the two ends attached to rigid supports (see Fig. 11). Call x_i the displacement of particle i from its steady state $(i = 1, \ldots, s)$; the Lagrangian may be written as

$$L = \tfrac{1}{2}m\sum_{i=1}^{s}\dot{x}_i^2 - \tfrac{1}{2}k\sum_{i=0}^{s}(x_{i+1} - x_i)^2$$

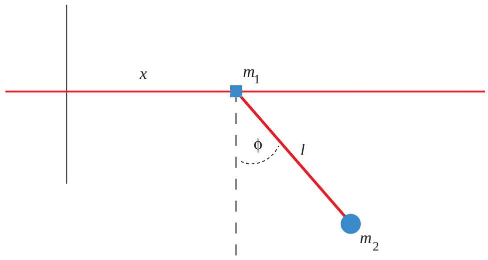

Figure 10: A horizontally sliding point of support (mass m_1) from which hangs a simple pendulum (mass m_2, length ℓ).

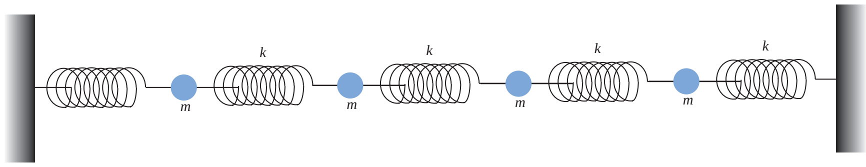

Figure 11: System of masses on springs.

where we define $x_0 = x_{s+1} = 0$. Notice that $x_{i+1} - x_i$ is the extension of the spring located between particles $i+1$ and i.

(a) Find the eigenfrequencies, ω_α, for this system.

[Hint: The determinant of the tri-diagonal, s by s matrix

$$D_s = \begin{vmatrix} \lambda & -1 & 0 & \dots & 0 \\ -1 & \lambda & -1 & \dots & 0 \\ 0 & -1 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{vmatrix} = \frac{\sin((s+1)\phi)}{\sin\phi}$$

where $\lambda = 2\cos\phi$.]

(b) Derive the result given in the hint above. [Hint: By expanding in minors, one finds that the determinant obeys the recursion relation $D_s = \lambda D_{s-1} - D_{s-2}$.]

---

9. Consider the system illustrated in Fig. 12: two identical simple pendula (mass m, length ℓ, gravitational acceleration g) coupled by a mass-less spring (stiffness k). The motion of the pendula is in the xz-plane and we use the generalized co-ordinates θ_1 and θ_2. The potential energy for the spring is $\frac{1}{2}k(d-d_0)^2$ where $d(\theta_1, \theta_2)$ is the distance between the masses and d_0 is the separation between the pivots. Note that the steady state position for the system is $\theta_{10} = \theta_{20} = 0$.

(a) Find the exact Lagrangian $L(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$, that is, without using the small oscillations approximation.

(b) Find the kinetic energy in the small oscillations approximation and express it as

$$T = \frac{1}{2}\sum_{i=1}^{2}\sum_{k=1}^{2} m_{ik}\dot\theta_i\dot\theta_k$$

Give explicit expressions for the constant coefficients m_{ik}.

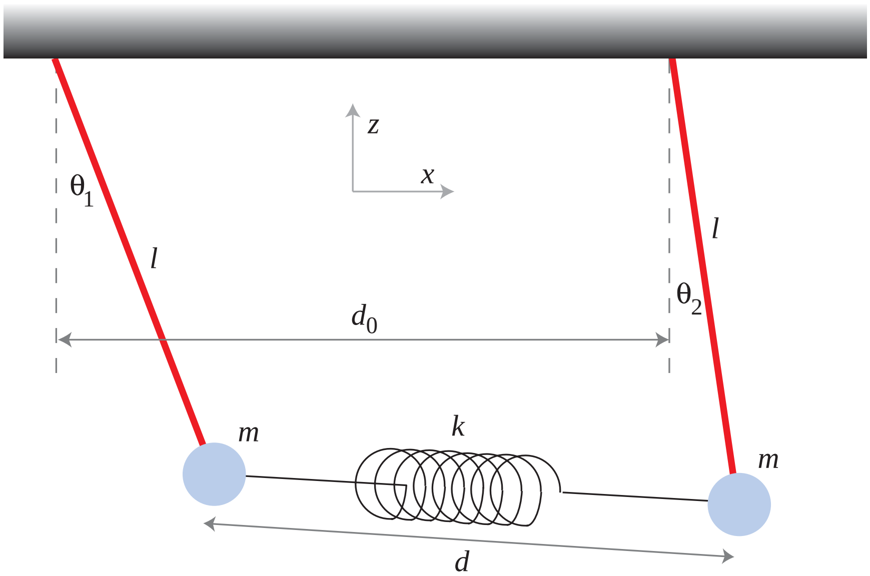

Figure 12: Two identical simple pendula (mass m, length ℓ, gravitational acceleration g) coupled by a mass-less spring (stiffness k).

(c) Find the potential energy in the small oscillations approximation and express it as

$$U = \frac{1}{2}\sum_{i=1}^{2}\sum_{k=1}^{2} k_{ik}\theta_i\theta_k$$

Give explicit expressions for the constant coefficients k_{ik}. [Hint: The small oscillations approximation in this case is the same as the small angle approximation so $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. Retain only quadratic terms in the potential; drop higher powers of θ. If you have any linear terms remaining in U, you've made a mistake in your algebra.]

(d) Find the two eigenfrequencies of the system in terms of m, ℓ, g, and k. Find the normal co-ordinates Q_1 and Q_2 in terms of θ_1 and θ_2.

---

10. Consider the system illustrated in Fig. 12: two identical simple pendula (mass m, length ℓ, gravitational acceleration g) coupled by a mass-less spring (stiffness k). The motion of the pendula is in the xz-plane and we use the generalized co-ordinates θ_1 and θ_2. In the small angle approximation the Lagrangian may be written as

$$L = \tfrac{1}{2}m\ell^2(\dot\theta_1^2 + \dot\theta_2^2) - \tfrac{1}{2}mg\ell(\theta_1^2 + \theta_2^2) - \tfrac{1}{2}k\ell^2(\theta_1 - \theta_2)^2$$

in the absence of friction (see previous exercise). We now consider the motion in the presence of frictional forces (e.g., imperfect pivot hinges) of the form $f_{\mathrm{fr},k} = -\alpha\dot\theta_k$ where $k = 1, 2$ and α is a positive coefficient.

(a) Using the trial solution $\theta_k = A_k \exp(rt)$, find r.

(b) Show that in the limit of large friction, as $t \to \infty$, the motion goes as $\exp(-(mg\ell/\alpha)t)$.

(c) Show that the result in part (b) is equivalent to the small oscillations of a *single* pendulum in the terminal velocity limit (i.e., negligible acceleration).

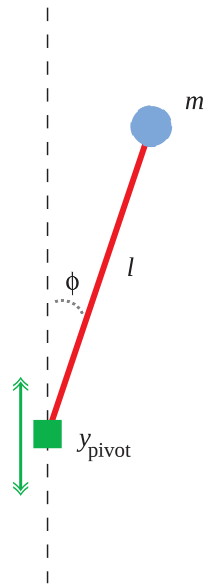

Figure 13: Inverted pendulum (mass m, length ℓ) with oscillating point of support ($y_{\text{pivot}}(t) = -a\cos\gamma t$).

---

11. Consider the driven, *inverted* pendulum (see Fig. 13). The mass m on a rigid, mass-less arm (length ℓ) is displaced from the maximum height by an angle ϕ. The point of support (i.e., the pivot) oscillates vertically, specifically, its vertical position is $y_{\text{pivot}}(t) = -a\cos\gamma t$; take the motion to be in the xy plane.

(a) Construct the Lagrangian for the inverted pendulum.

(b) From the Lagrangian obtain the equation of motion and show that in the small angle approximation it may be written as

$$\ddot{\phi} - \omega_0^2\phi\left[1 + \frac{a\gamma^2}{g}\cos\gamma t\right] = 0$$

where $\omega_0 = \sqrt{g/\ell}$.

(c) Show that the equation of motion above may be written as the Mathieu equation,

$$\ddot{\phi} + \phi\left[\alpha - 2\beta\cos 2t\right] = 0$$

and express α and β in terms of ω_0^2, γ, a, and ℓ.

(d) Figure 14, taken from Chapter 20 of Abramowitz and Stegun, shows the regions of stable solutions for the Mathieu equation (vertical axis is α, horizontal axis is $|\beta|$). Show that the driven inverted pendulum is *stable* for sufficiently high frequencies and give one example of values for γ and a (in terms of ω_0 and ℓ) that gives a stable solution. [Hint: Read the original caption.]

---

12. Consider a simple pendulum (mass m, length ℓ) in the "medium angle" approximation, that is, keeping terms up to order $\phi^{(3)}$.

(a) Find the equation of motion.

(b) Taking $\phi^{(1)} = a\cos\omega t$, where $\omega = \omega_0 + \omega^{(2)} + \ldots$ and $\omega_0 = \sqrt{g/\ell}$, show that the method of successive approximations gives $\phi^{(2)} = 0$.

(c) Find the leading order non-linear correction to the frequency, $\omega^{(2)}$ in terms of a and ω_0.

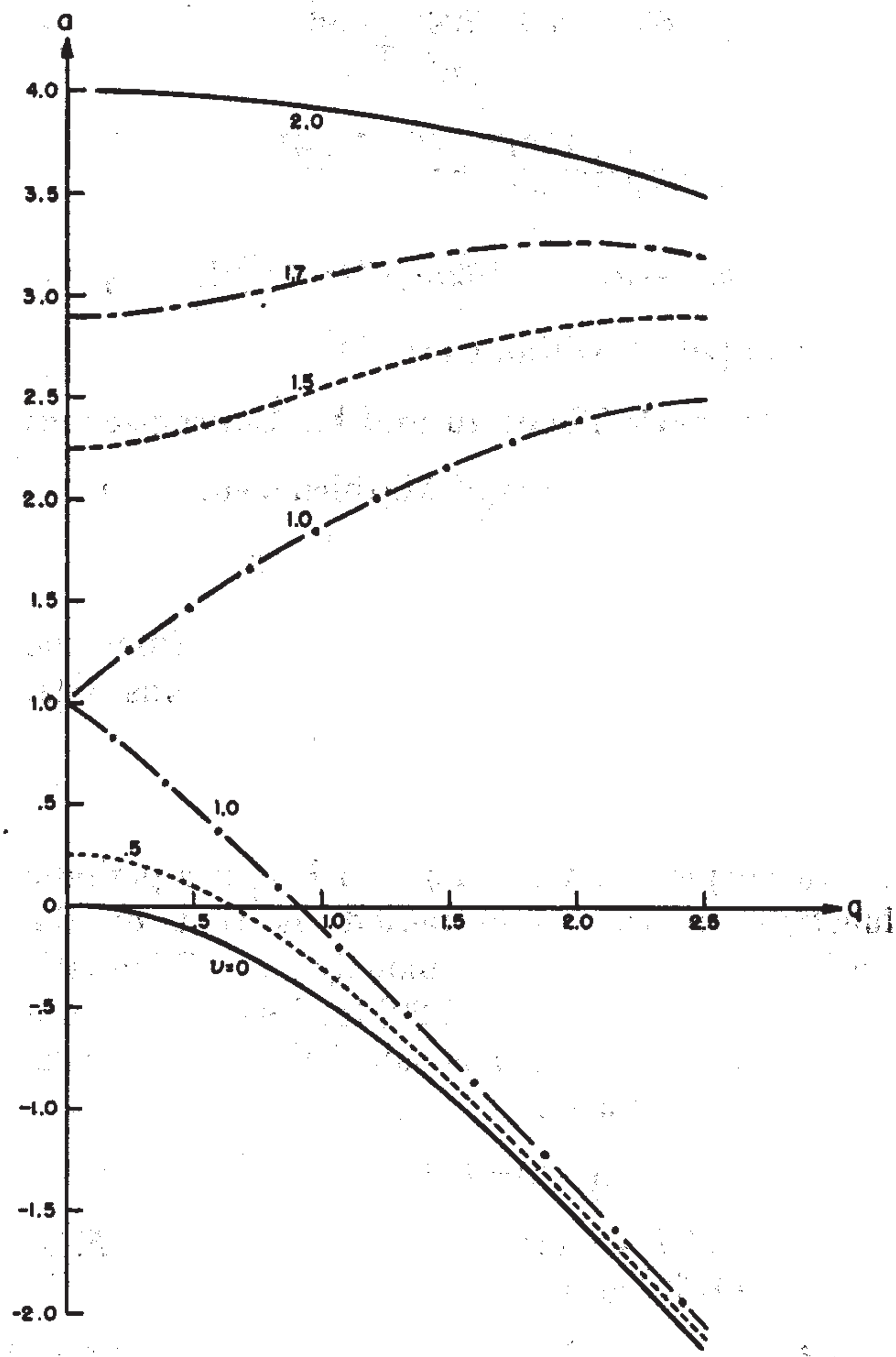

FIGURE 20.6. *Characteristic Exponent-First Two Stable Regions* $y=e^{i\nu x}P(x)$ *where* $P(x)$ *is a periodic function of period* π.

Definition of ν;

In first stable region, $0 \leq \nu \leq 1$,

In second stable region, $1 \leq \nu \leq 2$.

(Constructed from tabular values supplied by T. Tamir, Brooklyn Polytechnic Institute)

Figure 14: Regions of stable solutions for the Mathieu equation; from Abramowitz and Stegun.

(d) Find the leading order non-linear correction to the solution, $\phi^{(3)}$.

(e) Show that the result from part (c) is in agreement with the period of oscillation derived in section 11.

(f) Graph the small angle approximation $\phi_{\rm s} = a\cos\omega_0 t$, and the medium angle approximation $\phi_{\rm m} = \phi^{(1)} + \phi^{(3)}$ for $\omega_0 = 1$ and $a = \pi/2$ from $t = 0$ to $t = 6\pi$. Take $\omega \approx \omega_0 + \omega^{(2)}$.

---

13. Consider a system with the Lagrangian

$$L = \tfrac{1}{2}m\dot{q}^2 + Aq^2 - Bq^4$$

where m, A and B are positive constants.

(a) Find all the steady states of this system. Recall that if q_0 is a steady state then $\dot{q}(t) = 0$ for the initial conditions $q(0) = q_0$ and $\dot{q}(0) = 0$.

(b) From among the steady states choose one of the stable equilibria and find the frequency of small oscillations about that point.

(c) For the stable equilibrium point used in part (b), find the anharmonic correction, $\omega^{(2)}$, to the small oscillations frequency for the first approximation $x^{(1)} = a\cos\omega t$.

6 Motion of a Rigid Body

---

1. Consider a rigid body consisting of a massless cube of unit length with point masses at each of its corners (see Fig. 15). Six of these have mass m_a and the remaining two, which are on opposite corners, have mass m_b.

(a) Consider a coordinate system with its origin at one of the m_b point masses and axes along the edges of the cube. Find the inertial tensor of the body for this coordinate system.

(b) Suppose the cube is rotating with angular velocity $\mathbf{\Omega}$, where this vector points along the direction of the line connecting the two masses m_b. Find the kinetic energy in terms of m_a, m_b, and Ω; take the center of mass velocity, $\mathbf{V} = 0$.

(c) Use the parallel axis theorem to find the inertial tensor for a coordinate system with an origin at the cube's center and axes perpendicular to the cube's faces. [Hints: First show that the center of mass of the body is the geometric center of the cube. Check that if $m_a = m_b$ then the tensor is diagonal, that is, axes of the coordinate system are the principal axes.]

---

2. Consider a massless rigid rod of length ℓ with point masses m_a and m_b fixed at each end (call $\mu = m_a + m_b$); see Fig. 16. The principal axes of this body are parallel to the rod and two arbitrary directions perpendicular to the rod. We take our body-fixed co-ordinates to have an origin at the center of mass and axes along the principal axes; call $\hat{x}_3$ the axis parallel to the rod, pointing from the center of mass to mass m_b.

(a) Find the inertial tensor, in terms of the reduced mass $m = m_a m_b/\mu$ and ℓ, for the body-fixed coordinate system.

(b) Suppose that a torque is applied such that the body rotates with a constant angular velocity, $\mathbf{\Omega}$. In the body-fixed co-ordinate system the components are $\Omega_1 = \Omega \sin\theta$, $\Omega_2 = 0$, and $\Omega_3 = \Omega \cos\theta$,

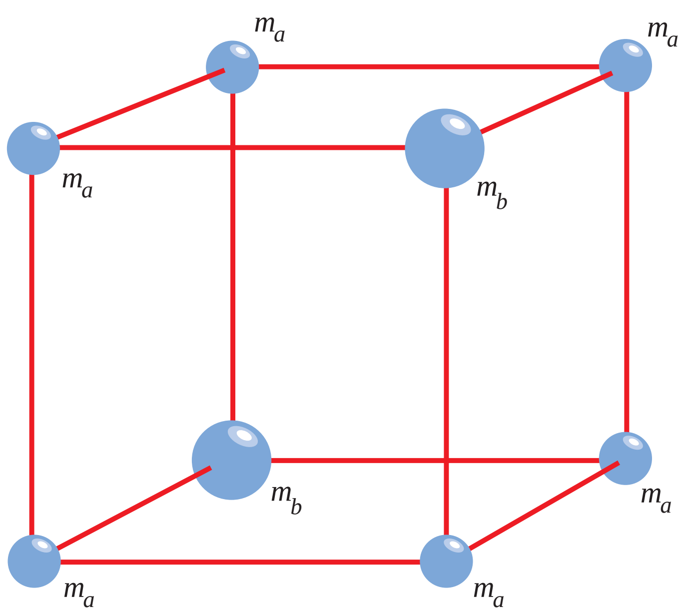

Figure 15: Massless cube of unit length with point masses its corners; six of mass m_a plus two, on opposite corners, of mass m_b.

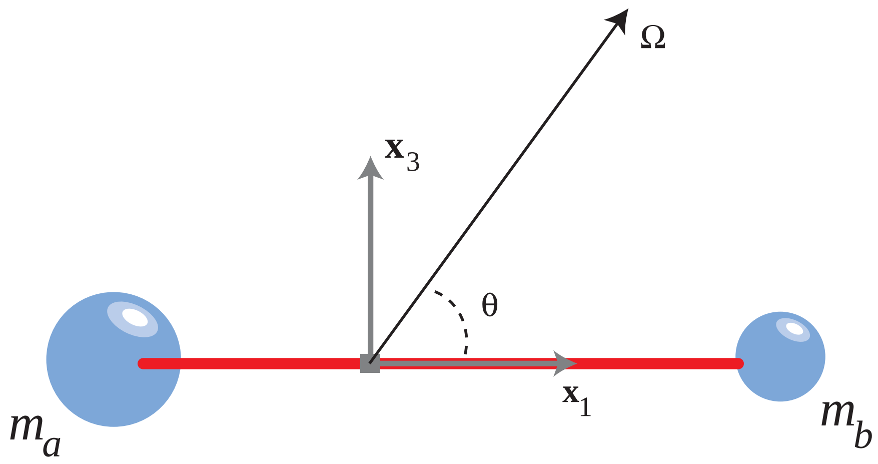

Figure 16: Massless rigid rod (length ℓ) with point masses (m_a, m_b) fixed at each end.

where $\Omega = |\mathbf{\Omega}|$ and θ are constants (see illustration below). Find the components of the applied torque, $\mathbf{K}$ in the body-fixed coordinate system.

(c) For the motion described in part (b), can the applied torque be zero? If yes, when does this occur?

---

3. Consider a solid cylinder (mass density ρ, length L, radius R_1) into which a cylindrical hole (radius R_2) is made. The hole is parallel to the cylinder's original (i.e., pre-hole) axis but displaced a distance d from the centerline; the hole is entirely within the cylinder (i.e., $d < R_1 - R_2$).

(a) Find the distance, a, from the center of mass to the cylinder's original axis.

(b) Find the rotational inertia, I, of the cylinder rotating about its original axis.

(c) Find the Lagrangian for the cylinder rolling on a horizontal surface in the presence of gravity. Take the rotation angle, ϕ, as your generalized coordinate with $\phi = 0$ when the center of mass is at its lowest point.

(d) Using the small oscillation approximation, find the period of the rocking motion of the cylinder on a horizontal surface.

---

4. Consider an asymmetric top ($I_1 < I_2 < I_3$) rotating freely (i.e., no external torques). (a) Using Euler's equations show that

$$\frac{d\Omega_2}{dt} = \frac{1}{I_2\sqrt{I_1 I_3}}\sqrt{I_2(I_2 - I_1)\Omega_2^2 - (M^2 - 2I_1 T)}\sqrt{(M^2 - 2I_3 T) - I_2(I_2 - I_3)\Omega_2^2}$$

where

$$T = \tfrac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2), \qquad M = \sqrt{I_1^2\Omega_1^2 + I_2^2\Omega_2^2 + I_3^2\Omega_3^2}$$

are the kinetic energy and the magnitude of the angular momentum. (b) Show that if $M^2 = 2TI_2$ then

$$\Omega_2(t) = \Omega_\infty \tanh(t/\tau)$$

where $\Omega_\infty = 2T/M$ and

$$\tau = \frac{1}{\Omega_\infty}\sqrt{\frac{I_1 I_3}{(I_3 - I_2)(I_2 - I_1)}}$$

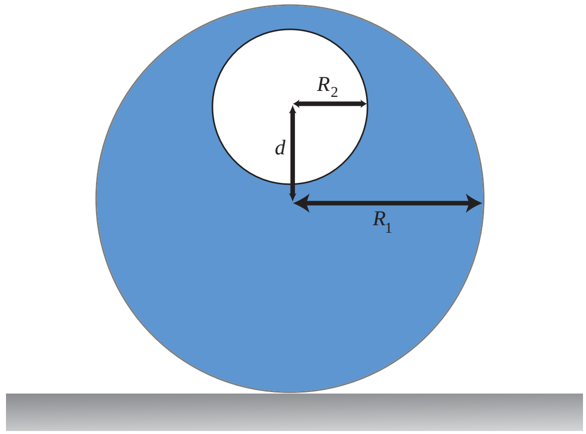

Figure 17: A solid cylinder (mass density ρ, length L, radius R_1) with a cylindrical hole (radius R_2) a distance d from the centerline.

with the initial condition $\Omega_2(0) = 0$. Note that M and T are constants of the motion (i.e., conservation of angular momentum and energy). [Hint: Simplify the equation you derived in part (a) and then substitute in the expression given for $\Omega_2(t)$ to check that it obeys the equation.]

(c) Use the results from parts (a) and (b) to obtain an expression for $\Omega_1(t)$ in terms of Ω_∞, τ, and the three components of the inertial tensor. Make a sketch showing the graph of $\Omega_1(t)$ and $\Omega_2(t)$ versus time.

---

5. Consider the torque-free motion of a slightly asymmetric top with $I_1 + \Delta I = I_2 = I_3 - \Delta I$ where $|\Delta I| \ll I_2$. Take the axes of the "body" co-ordinate system, x_1, x_2, and x_3, to point along the principal axes of the body.

(a) Suppose that the angular velocity, $\mathbf{\Omega}(t)$, points approximately in the x_3 direction, specifically take $\Omega_1(t) = \epsilon_1(t)$, $\Omega_2(t) = \epsilon_2(t)$, and $\Omega_3(t) = \omega + \epsilon_3(t)$ where ω is a constant and $|\boldsymbol{\epsilon}| \ll |\mathbf{\Omega}|$. Find $\epsilon_1(t)$; take the initial conditions $\epsilon_1(0) = A$ and $\epsilon_2(0) = 0$.

(b) Repeat your analysis in part (a) but with the angular velocity pointing approximately in the x_2 direction, specifically take $\Omega_1(t) = \epsilon_1(t)$, $\Omega_2(t) = \omega + \epsilon_2(t)$, and $\Omega_3(t) = \epsilon_3(t)$ where ω is a constant and $|\boldsymbol{\epsilon}| \ll |\mathbf{\Omega}|$. Find $\epsilon_1(t)$, taking the initial conditions $\epsilon_1(0) = A$ and $\epsilon_3(0) = 0$, and show that it increases exponentially with time.

(c) Suppose that the angular velocity, $\mathbf{\Omega}(t)$, points approximately in the x_3 direction (as in part (a)). Show that in describing $\mathbf{\Omega}(t)$ we may take $\epsilon_3(t) \approx$ constant given that $|\boldsymbol{\epsilon}| \ll |\mathbf{\Omega}|$ (i.e., $\mathbf{\Omega}$ changes primarily because ϵ_1 and ϵ_2 change).

---

6. Consider a homogeneous circular cone (mass μ, height h, base radius R), that lies on an incline plane, which makes an angle α with the horizontal. The cone's vertex angle is 2α so when the cone is at rest its axis is horizontal (see Fig. 18). The cone rolls without slipping in the presence of a gravitational acceleration g. Find the small oscillations frequency for the motion of the cone rocking back and forth from its rest position.

[Hint: When the cone is near its rest position the height of a point, located on the cone's axis and a distance a from the vertex, is

$$z(s) \approx \frac{1}{2}\frac{a}{h^2}\sin\alpha\cos^3\alpha\ s^2$$

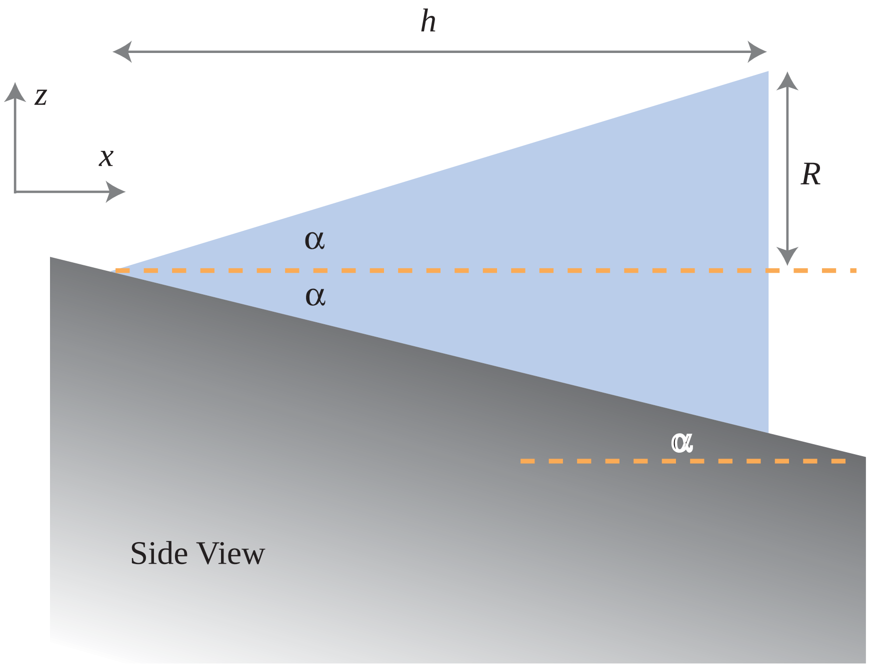

Figure 18: Homogeneous circular cone (mass μ, height h, base radius R, vertex angle 2α) that lies on an incline plane, which makes an angle α with the horizontal.

where s is the arc length distance that the circular base has rolled along the surface.]

---

7. Consider inertial and rotating ("body") frames of reference with a common origin, with the latter rotating with constant angular velocity $\boldsymbol{\Omega}$ relative to the former.

(a) Show that,

$$\ddot{\mathbf{r}}_b = \ddot{\mathbf{r}}_i - 2\boldsymbol{\Omega} \times \dot{\mathbf{r}}_b - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$$

where

$$\ddot{\mathbf{r}}_b \equiv \left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\text{body}} \qquad \ddot{\mathbf{r}}_i \equiv \left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\text{inertial}}$$

Note that Landau (see (36.1)) uses the more compact notation $d'\mathbf{A}/dt = (d\mathbf{A}/dt)_{\text{body}}$.

(b) The Lorenz force acting on a particle (mass m, charge e) is

$$\mathbf{F} = e(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$$

where E and B are the applied electric and magnetic fields. Show that for a weak uniform magnetic field the motion of a charged particle in a central electric field may be expressed as

$$\ddot{\mathbf{r}}_b \approx \frac{e}{m}\mathbf{E}$$

where $\mathbf{r}_b$ is the position of the particle in a coordinate system rotating with angular velocity $\boldsymbol{\Omega} = -(e/2m)\mathbf{B}$; give an explicit expression for how weak $|B|$ should be, in terms of e, m, $|\mathbf{E}|$, and $|\mathbf{r}|$ for this approximation to be valid.

(c) Show that the result in part (b) may be extended to a system of particles (each of mass m, charge e) in a central electric field and weak uniform magnetic field given that the particles interact through a potential that only depends on their separations (i.e., $U = \sum_\alpha \sum_\beta U_{\alpha\beta}(|\mathbf{r}_\alpha - \mathbf{r}_\beta|)$).

7 The Canonical Equations

---

1. Consider a point mass m moving in the x-y plane subject to a constant gravitational acceleration g; its Hamiltonian is

$$H(x, y, p_x, p_y) = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + mgy$$

so our canonical coordinates and momenta are $q_1 = x$, $q_2 = y$, $p_1 = p_x$, and $p_2 = p_y$.

(a) Solve the Hamilton-Jacobi equation and show that the action may be written as,

$$S = \sqrt{2m\alpha_1}\, x - \frac{1}{3m^2 g}(2m\alpha_2 - 2m^2 gy)^{3/2} - (\alpha_1 + \alpha_2)t + S_0'$$

where S_0' is a constant. [Hint: Use the separated trial solution $S = S_x(x) + S_y(y) - \alpha t + S_0$; note, $\alpha \neq \alpha_1 \neq \alpha_2$, $S_0 \neq S_0'$.].

(b) As discussed on page 148 (fourth paragraph), we are free to take the constants $P_1 = \alpha_1$, $P_2 = \alpha_2$ as our new momenta and the constants $Q_1 = \beta_1$, $Q_2 = \beta_2$ as our new co-ordinates. Use

$$Q_1 = \frac{\partial S}{\partial P_1} \qquad Q_2 = \frac{\partial S}{\partial P_2}$$

to obtain explicit expressions for $x(t)$ and $y(t)$ in terms of α_1, α_2, β_1, β_2, m, and g.

(c) Fix the constants of integration in your expressions from part (b) by using the initial conditions

$$x(0) = x_0 \qquad \dot{x}(0) = v_{x_0} \qquad y(0) = y_0 \qquad \dot{y}(0) = v_{y_0}$$

to find explicit expressions for $x(t)$ and $y(t)$ for these initial conditions.

---

2. Consider a particle (mass m) moving in the horizontal plane under the constraint $r(t) = r_0(1 - t/\tau)$ so the initial radial position is r_0 and the particle is pulled into the origin at time τ. You can imagine this as a particle on a frictionless table attached to a string that is being pulled into a hole in the center of the table (see Fig. 19) Using polar co-ordinates the initial angle is $\phi(0) = 0$ and the initial angular velocity is $\dot{\phi}(0) = \omega_0$.

(a) Write the Lagrangian and use Lagrange's equation to obtain an explicit expression for $\phi(t)$ in terms of ω_0, τ and t. [Hint:

$$\int \frac{dx}{(a + bx)^2} = \frac{-1}{b(a + bx)}$$

where a and b are constants.] [15 points]

(b) Write the Hamiltonian and use the canonical equations to obtain an explicit expression for $\phi(t)$ in terms of ω_0, τ and t. [10 points]

---

3. Consider a simple spherical pendulum, that is, a point of mass m constrained to move on the surface of a sphere of radius R in a constant gravitational field with constant downward acceleration $\mathbf{g} = -g\hat{z}$.

(a) Find the Hamiltonian, $H(\theta, \phi, p_\theta, p_\phi)$, in spherical coordinates.

(b) Show that p_ϕ is a constant.

(c) Show that p_θ is a constant if $\theta = \theta_0$ where θ_0 is obtained from the equation

$$p_\phi^2 = A \sin^3\theta_0 \tan\theta_0$$

where A is a constant. Do not bother solving explicitly for θ_0 (messy but straight-forward).

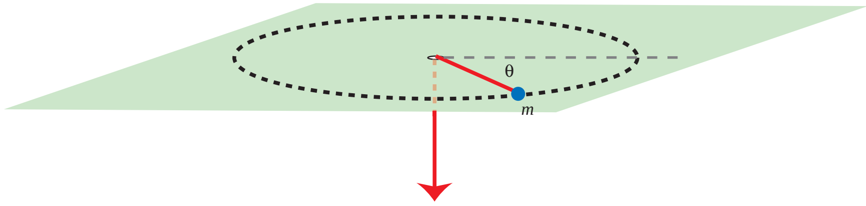

Figure 19: Particle on a frictionless table attached to a string that is being pulled into a hole in the center of the table.

(d) Expand the Hamiltonian $H(\theta_0 + \theta_1, \phi, p_\theta, p_\phi)$ about θ_0 to quadratic order in θ_1. Show that $H = H_0 + H_2\theta_1^2 + O(\theta_1^3)$ where H_0 and H_2 are constants. Obtain explicit expressions for H_0 and H_2; show that your result for H_2 may be written as

$$H_2 = B\frac{3\cos^2\theta_0 + 1}{\cos\theta_0}$$

where B is a constant. [Hint: You do not need an expression for θ_0, just take it as a constant.]

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4. Consider the transformation

$$Q = \frac{p}{\tan q} \qquad\qquad P = \log\left(\frac{\sin q}{p}\right)$$

(a) Use Poisson brackets to show that this transformation is canonical.
(b) Express this transformation as $p(q, Q)$ and $P(q, Q)$.
(c) Finding the corresponding generating function $F(q, Q)$ for this transformation. [Hint: The condition $p = \partial F/\partial q$ implies

$$F(q, Q) = \int p(q, Q)\, dq + a(Q)$$

where $a(Q)$ is an unspecified function of only Q.]
(d) Check your generating function by showing that $p = \partial F/\partial q$ and $P = -\partial F/\partial Q$.
(e) Find the generating function $\Phi(q, P)$; check your result by showing that $p = \partial\Phi/\partial q$ and $Q = \partial\Phi/\partial P$.

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5. In Cartesian co-ordinates, the relativistic Hamiltonian for a free particle (i.e., no potential energy) is

$$H(x, y, z, p_x, p_y, p_z) = c\sqrt{m^2c^2 + p_x^2 + p_y^2 + p_z^2}$$

where m is the particle's mass and c is the speed of light.
(a) Solve the Hamilton-Jacobi equation and find the function $S(x, y, z, t)$.
(b) From your result, obtain $x(t)$. Fix the constants by the initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$.
(c) From your results, obtain an expression for the energy in terms of m, c, and the magnitude of the velocity.

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6. Consider the Hamiltonian

$$H(q_1, q_2, p_1, p_2) = q_1p_1 - q_2p_2 - aq_1^2 + bq_2^2$$

where a, b are constants.

(a) Show that $f_a = q_1 q_2$ is an integral of the motion, that is, $df_a/dt = 0$.

(b) Is $f_b = q_1 e^{-t}$ an integral of the motion? Justify your answer.

(c) Show that $f_c = q_1^\alpha (p_2 + \beta q_2)$ is an integral of the motion for certain values of α and β and find specific expressions for these constants.

(d) Solve the canonical equations and explicitly verify that the above are integrals of motion.

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7. In polar co-ordinates, Hamiltonian for the relativistic Kepler problem may be written as

$$H = c\sqrt{p_r^2 + \frac{p_\phi^2}{r^2} + m^2c^2} - \frac{k}{r}$$

(a) Solve the Hamilton-Jacobi equation to find the action $S(r, \phi, t)$. You may leave your expressions in terms of integrals.

(b) From the action, obtain a relation between ϕ and r of the form $\phi = \int f(r)dr + \phi_0$ where ϕ_0 is a constant.

Miscellaneous Exercises

1. A particle of mass m can slide along a wire AB whose perpendicular distance to the origin is h (see Fig. 20). The line OC rotates about the origin at a constant angular velocity $\dot{\theta} = \omega$ with $\theta = 0$ at $t = 0$. The particle is subject to a constant gravitational force of magnitude mg in the $-y$ direction.

(a) Taking the distance q between the particle and the point C as the generalized coordinate, find the Lagrangian. Express this Lagrangian in its simplest form, discarding all unnecessary terms.

(b) Is the energy conserved in this system? Justify your answer.

(c) Find the equation of motion.

(d) Solve the equation of motion and find $q(t)$ for the initial conditions $q(0) = \dot{q}(0) = 0$.

(e) Show that when $t \ll \omega^{-1}$ the solution from part (d) is $q(t) \approx \frac{1}{2}gt^2$.

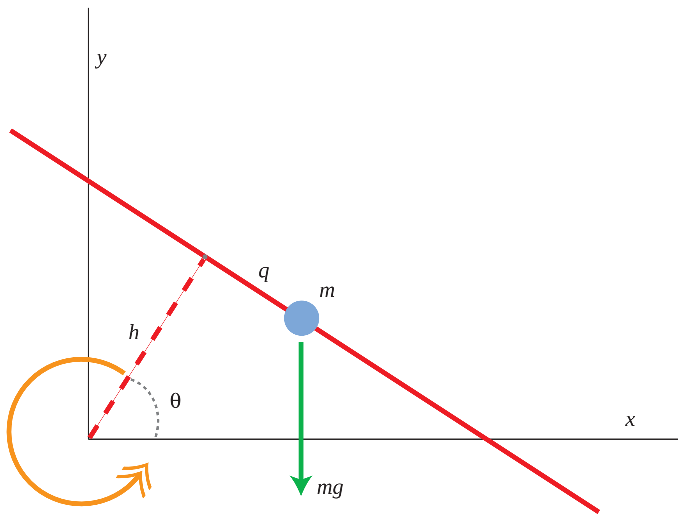

Figure 20: Particle (mass m) sliding on a wire AB whose perpendicular distance to the origin is h; the line OC rotates about the origin as $\dot{\theta} = \omega$ with $\theta(0) = 0$.

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