Homework 11 (Due Tuesday, March 13th)

1. Graph the fermi integral $f_{3/2}(z)$ from $z = 0.1$ to 100 on a loglog scale. On the same plot graph the small z approximation $f_{3/2}(z) \approx z$ and the large z approximation $f_{3/2}(z) \approx 4/(3\sqrt{\pi})(\ln z)^{3/2}$.

2. The chemical potential at $T = 0$ equals the Fermi energy. Obtain μ in the low temperature limit (but for $T > 0$) by keeping the next order in the expansion for $f_{3/2}(z)$. Specifically, use

$$f_{3/2}(z) = \frac{4}{3\sqrt{\pi}} (\log z)^{3/2} \left[1 + \frac{\pi^2}{8} (\log z)^{-2} + \ldots \right]$$

to show that

$$\mu \approx \epsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{kT}{\epsilon_F} \right)^2 \right]$$

where ϵ_F is the Fermi energy.

3. In the low temperature limit the Fermi equation of state is,

$$P = \frac{2}{5} \epsilon_F \frac{N}{V} \left[1 + \frac{5\pi^2}{12} \left(\frac{kT}{\epsilon_F} \right)^2 + \ldots \right]$$

Show that the Helmholtz free energy for the ideal Fermi gas at low temperatures is

$$A \approx a_1 N \epsilon_F \left[1 - a_2 \left(\frac{kT}{\epsilon_F} \right)^2 \right]$$

where a_1 and a_2 are constant, positive coefficients, which you need to find. [Hint: Use $dA = -PdV - SdT$ and integrate on an isotherm; fix the constant of integration by the value at $T = 0$.]

4 (a) Show that for an ideal Fermi gas,

$$\gamma = \frac{C_P}{C_V} = \frac{(\partial z/\partial T)_{P,N}}{(\partial z/\partial T)_{V,N}}$$

Note that this result holds also holds for Bose quantum gases above the critical temperature.

(b) Show that for an ideal Fermi gas,

$$\frac{1}{z} \left(\frac{\partial z}{\partial T} \right)_{P,N} = -\frac{5}{2T} \frac{f_{5/2}(z)}{f_{3/2}(z)}$$

Note that the same result holds for a Bose gas with the functions $f_{n/2}$ replaced with $g_{n/2}$.

(c) Show that for an ideal Fermi gas,

$$\frac{1}{z} \left(\frac{\partial z}{\partial T} \right)_{V,N} = -\frac{3}{2T} \frac{f_{3/2}(z)}{f_{1/2}(z)}$$

Note that the same result holds for a Bose gas with the functions $f_{n/2}$ replaced with $g_{n/2}$.

(d) Show that in the low temperature limit the ratio of heat capacities in a Fermi gas is

$$\gamma \approx 1 + A \left(\frac{kT}{\epsilon_F} \right)^2$$

and find the numerical constant, A.