Homework 22 (Due Thursday, May 10th)

For the problems below you'll be computing means and variances for fluid properties in a volume element by evaluating averages of the form

$$\langle X \rangle = \sum_{N=0}^{\infty} \int d\mathbf{v}_1 \ldots \int d\mathbf{v}_N X(N, \mathbf{v}_1, \ldots, \mathbf{v}_N) P_{\scriptscriptstyle N}(N) P_v(\mathbf{v}_1) \ldots P_v(\mathbf{v}_N)$$

The distribution of the velocities, $P_v(\mathbf{v})$, for classical particles is the Maxwell-Boltzmann distribution with mean and variance,

$$\begin{aligned} \langle \mathbf{v} \rangle &= \int d\mathbf{v}\, \mathbf{v} P_v(\mathbf{v}) = \overline{\mathbf{u}} \\ \langle |\delta \mathbf{v}|^2 \rangle &= \int d\mathbf{v}\, |\mathbf{v} - \langle \mathbf{v} \rangle|^2 P_v(\mathbf{v}) = \frac{3\, k_B T}{m} \end{aligned}$$

where $\overline{\mathbf{u}}$ is the center-of-mass velocity of the particles. Note that thermodynamic equilibrium does *not* imply $\overline{\mathbf{u}} = 0$ since a system is in equilibrium in all inertial frames of reference.

The distribution for the number of particles, $P_{\scriptscriptstyle N}(N)$, depends on the equation of state for the fluid. For the present analysis we take the mean and the variance,

$$\begin{aligned} \langle N \rangle &= \sum_{N=0}^{\infty} N P_{\scriptscriptstyle N}(N) = \overline{N} \\ \langle \delta N^2 \rangle &= \sum_{N=0}^{\infty} (N - \langle N \rangle)^2 P_{\scriptscriptstyle N}(N) = \sigma_N^2 \end{aligned}$$

as given. In dense fluids σ_N^2 is small since it is proportional to the fluids' compressibility; in the case of a dilute gas, N is Poisson distributed with $\sigma_N^2 = \overline{N}$.

1. Define the instantaneous fluid velocity as the average velocity of the particles,

$$\hat{\mathbf{u}} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{v}_i$$

(a) Show that $\langle \hat{\mathbf{u}} \rangle = \overline{\mathbf{u}}$. Note that you need to exclude the $N = 0$ case since it is meaningless to assign an instantaneous velocity when there are no particles. (b) Show that for the variance of instantaneous fluid velocity,

$$\langle |\hat{\mathbf{u}} - \langle \hat{\mathbf{u}} \rangle|^2 \rangle = \frac{3\, k_B T}{m \overline{N}} \left(1 + \frac{\sigma_N^2}{\overline{N}^2} + \ldots \right)$$

Again, be careful with the case where $N = 0$.

2. One may define an instantaneous temperature as,

$$\hat{T} = \frac{1}{\frac{3}{2} k_B N} \left(K - \frac{1}{2} m N |\hat{\mathbf{u}}|^2 \right) \qquad \text{where} \qquad K = \sum_{i=1}^{N} \frac{1}{2} m |\mathbf{v}_i|^2 \qquad \text{and} \qquad \hat{\mathbf{u}} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{v}_i$$

(a) Explain and justify this expression for temperature. (b) Show that $\langle \hat{T} \rangle \neq T$. As in the previous problem, be careful with how you treat the $N = 0$ case. (c) Show that the alternative definition,

$$\hat{T}' = \frac{1}{\frac{3}{2} k_B (N-1)} \left(K - \frac{1}{2} m N |\hat{\mathbf{u}}|^2 \right)$$

gives $\langle \hat{T}' \rangle = T$ (be careful with both the $N = 0$ and $N = 1$ cases). (d) Explain and justify this new expression for the temperature, specifically the physical meaning behind the change from N to $N - 1$ in the second definition.