Homework 5 (Due Tuesday, February 14th)

1. Consider a system that has Ω states; the probability of the system being in a particular state is $\mathcal{P}_i$. The entropy is defined as

$$S = -k \sum_{i=1}^{\Omega} \mathcal{P}_i \ln \mathcal{P}_i$$

Show that the entropy is maximum when $\mathcal{P}_i = 1/\Omega$ using the method of Lagrange multipliers to impose the condition that the probabilities sum to one.

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2. Write a program to explicitly count the number of states that satisfy the condition $n_x^2+n_y^2+n_z^2 \leq E^*$ where $n_i = 1, 2, \ldots$. This gives Σ, the number of states at or below the dimensionless energy $E^* = 8mL^2U/h^2$ for a quantum mechanical particle in a cubic box. Graph the number of states versus E^* (from $E^* = 1$ to 10^3) on a log-log scale.

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3. Demonstrate that the number of ways one can place M indistinguishable objects into N boxes is

$$\binom{M+N-1}{N-1} = \frac{(M+N-1)!}{(N-1)!M!}$$

The clearest and easiest way to formulate this demonstration is graphically rather than analytically.

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4. Consider a one-dimensional chain with $N \gg 1$ segments, as illustrated below. Let the length of each segment be a when the segment is horizontal and zero when the segment is vertical (these are the only two possible states for a segment). Call L the total length of the chain.

(a) Find the entropy of this system as a function of L. Hint: The number of combinations for choosing M objects out of N is $N!/[M!(N-M)!]$.

(b) Suppose that the chain is under a tension F at temperature T. Find $L(T, F)$. [Hints: $F = -dU/dL$ and you'll want to use Sterling's approximation]

(c) Show that at high temperature the length of the chain is linearly proportional to the tension (i.e., Hooke's law).