

Program 30 / Chapter 40

Lorentz transformations in special relativity

At high speeds (near the speed of light) the Newtonian laws of mechanics must be corrected according to Einstein's theory of special relativity. While it is a simple theory mathematically, special relativity is extremely unintuitive—read and reread Chapter 40 of your textbook very carefully. In this section we consider the chronological sequence of events as determined by different observers moving relative to each other.

Theory An event is described by its location in space and the time the event occurs; these values will differ for observers in different frames of reference. Consider an unprimed frame of reference, F , and a primed frame of reference, F' , the latter moving at velocity u in the x direction (see section 40–6 of your textbook). The origins of the two frames overlap at time 0 (in each frame). An event that occurs in frame F at location x and time t occurs in frame F' at location and time,

$$x' = \gamma (x - ut)$$

$$t' = \gamma \left(t - \frac{ux}{c^2} \right)$$

where

$$\gamma = \frac{1}{\sqrt{1 - (u/c)^2}}$$

The above transformations of space and time coordinates into the moving frame of reference are known as Lorentz transformations (see equations (40–20) to (40–22) in your textbook).

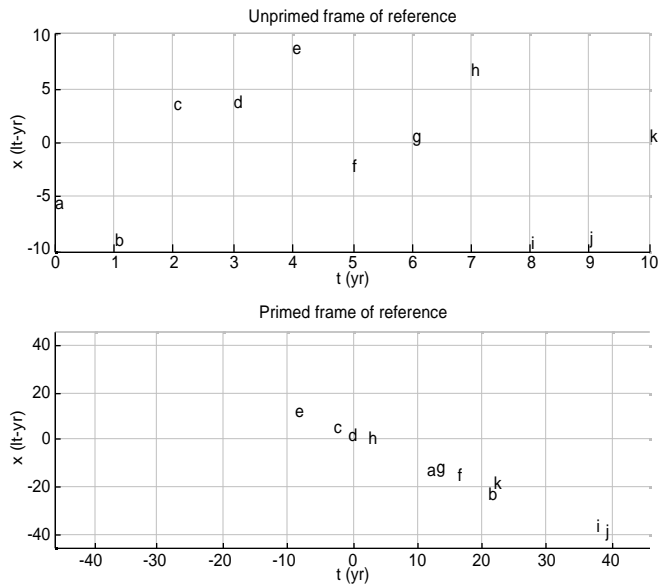
An interesting consequence of the Lorentz transformations is that the chronological sequence of several events can be different in different frames. In other words, if in frame F event “a” occurs before event “b”, in frame F' it can happen that event “b” occurs first. This seemingly paradoxical result does not imply that someone can be born before her mother is born, this reversal of chronological sequence can only occur when events are separated by space and time such that a light signal from one event cannot reach the other (i.e., such events cannot communicate between each other).

Program The MATLAB program `lortran`, which computes the time and place of events in two frames of reference using Lorentz transformations, is outlined below:

- Initialize variables (e.g., speed of moving frame).
- Set up graphics for plotting the events.
- Loop over desired number of events.
 - Compute the time and place of event in unprimed frame of reference.
 - Use Lorentz transformation to compute the time and place of the same event in the primed frame of reference.
 - Mark the event on the t - x graphs for each frame of reference.

The geometry used by the program is illustrated in Figure 40–18 of your textbook. In the unprimed frame, F , the events are evenly spaced in time but randomly distributed in space (imagine a galaxy-wide lottery held once a year with the event being holding the winning ticket). The program uses the Lorentz transformations to determine where and when each event occurs in the primed frame of reference F' .

With the moving frame of reference, F' , traveling at a speed of $0.9 c$, the program produces the graphs shown below.



Notice that in frame F event “a” occurs before event “e” while in the moving frame F' event “e” occurs first. Since the event locations are selected using a random number generator, you will get a different graph each time you run the program (without quitting MATLAB).

Exercises

1. (I) Run the `lortran` program and print out the resulting graphs for the following velocities of the primed frame of reference: (a) 0 ; (b) $0.1 c$; (c) $0.5 c$; (d) $0.9 c$ and (e) $0.99 c$.
2. (I) Using pencil and paper, find a specific example of two events for which $t_1 < t_2$ but $t'_1 > t'_2$, in other words, in the unprimed frame event 1 occurs before event 2 while in the primed frame event 2 occurs first. Give both the time and location of each event in both frames of reference.
3. (II) (a) Modify the `lortran` program to have the events always occur at $x = 0$. Run your program and print out the resulting graphs for the cases given in exercise 1. (b) Modify the `lortran` program to have the events occur at $x = ct$. Run your program and print out the resulting graphs for the cases given in exercise 1.
4. (II) Modify the `lortran` program to have it write the quantity $x^2 - c^2t^2$ for each event in each frame of reference. Run your program and print out the resulting values for the cases given in exercise 1.
5. (III) (a) Modify the `lortran` program to have the direction of the velocity of the primed frame of reference change after half the events have occurred. Run your program

and print out the resulting graphs for the cases given in exercise 1. (b) Repeat part (a) but have the events always occur at $x = 0$. This simulates the twin paradox, the events representing the birthdays of the twin that stays home.

6. (III) Modify the `lortran` program to have two events at each time in the unprimed frame and make these two events be the locations of two objects, one with twice the mass of the other, moving towards each other with a relative speed of $0.5c$. Have the objects collide elastically at the origin at a time equal to half the total time graphed. Run your program and print out the resulting graphs for the cases given in exercise 1.

Listing

`lortran.m`

```
% lortran - Program to compute time and place of events in two
%           frames of reference using Lorentz transformations
clear all; help lortran; % Clear memory; print header

%@ Initialize variables (e.g., speed of moving frame)
u = input('Enter velocity (as fraction of c): ');
gamma = 1/sqrt(1-u^2);
tMax = 10; % In unprimed frame, time goes from 0 to tMax
xMax = tMax; % In unprimed frame, events occur between -xMax and xMax

%@ Set up graphics for plotting the events
clf; figure(gcf); % Clear figure window and bring it forward
FirstChar = abs('a'); % First character for marking events is 'a'
subplot(2,1,1) % First (top) plot in 2x1 tile of graphics
    axis([0, tMax, -xMax, xMax]);
    grid on; % Place a hash grid on the graph
    xlabel('t (yr)'); % Horizontal axis label
    ylabel('x (lt-yr)'); % Vertical axis label
    title('Unprimed frame of reference');
    hold on; % Hold the graph as events are added
subplot(2,1,2) % Second (bottom) plot in 2x1 tile of graphics
    axis(2*gamma*[-tMax, tMax, -xMax, xMax]);
    grid on; % Place a hash grid on the graph
    xlabel('t (yr)'); % Horizontal axis label
    ylabel('x (lt-yr)'); % Vertical axis label
    title('Primed frame of reference');
    hold on; % Hold the graph as events are added

%@ Loop over desired number of events
NEvents = tMax+1;
for iEvent = 1:NEvents

    %@ Compute the time and place of event in unprimed frame
    x = (2*rand(1)-1)*xMax; % Event occurs at random location
    t = (iEvent-1)/(NEvents-1)*tMax; % Events uniformly spaced in time

    %@ Use Lorentz transformation to compute the time and place
    %% of the same event in primed frame
    xp = gamma*(x - u*t);
    tp = gamma*(t - u*x);

    %@ Mark the event on the t-x graphs for each frame of reference
    EventName = setstr(iEvent-1+FirstChar); % Event names are a,b,c,...
    subplot(2,1,1) % Top plot (unprimed frame)
    h = text(t,x,EventName);
    subplot(2,1,2) % Bottom plot (primed frame)
```

```
h = text(tp, xp, EventName);  
end  
drawnow; % Draw the graphs
```