Supplementary Exercises for
Numerical Methods for Physics

Alejandro L. Garcia

January 26, 2006

New exercises are listed as (Chapter).(Section).**X**(Number). Augmented exercises are listed as (Chapter).(Section).(Number)+. Please send comments and suggestions to **algarcia@algarcia.org**.

Chapter 1

1.2.X1 Consider the function

$$r(\theta) = \frac{1-\epsilon^2}{1-\epsilon\cos(\theta)}$$

a) Plot r versus θ from $\theta = 0$ to 2π for $\epsilon = \{0, 0.1, 0.5, 0.9\}$. [Computer] b) Plot this function in x, y coordinates taking r as the radial distance from the origin at angle θ (i.e., as polar coordinates). What is this function? [Computer]

1.2.X2 The function $f(x, y) = e^x \cos y$ from $x = -2$ to 1 and from $y = -2\pi$ to 2π is graphed in Fig. 1 by: a) contour plot; b) surface plot; c) mesh plot; d) combination surface and contour plot with a value scale on the side. Recreate each of these plots; don't forget axis labels and titles. [Computer]

1.4.X1 The purpose of this exercise is to get you started running the programs in the book on your computer and printing results on your printer. a) Print a copy of the program `orthog`. b) Run `orthog` for a variety of cases and print out the results. c) Run the `interp` program and print the resulting graph as shown in the textbook figure 1.2. [Computer]

1.5.X1 Take the variables $a = x + 1$, $b = a/x$, $c = b - 1$, $d = cx$, and $e = 1 - d$. a) Show that $e = 0$. [Pencil] b) Write a program that computes these variables and prints the value of e for $x = 1, \ldots, 10$. What is significant about the resulting values? [Computer] c) Modify your program from the previous part to plot e versus x for $x = 1, \ldots, 200$; use a different symbol for odd and even values of x. Comment on the results.

1.5.X2 Write a program to graph the polynomial $y = (x-1)^9$ but compute it using its expanded expression, $y = x^9 - 9x^8 + \ldots$. Plot $y(x)$ in the range $x = 0.96$ to 1.04 and comment on the results.

1.5.X3 Consider the quadratic $ax^2 + bx + c = 0$ with roots x_+, x_- (subscript given by $\pm$ choice in quadratic equation). a) Show that $x_+x_- = c/a$. [Pencil]. b) Write a program to compute the two roots using the quadratic equation and using a log-log scale plot $|a/c - x_+x_-|$ versus b for $b = 10^i$, $i = 1, \ldots, 15$. Comment on the results. [Computer] c) Repeat part b) for $b = 2^i$, $i = 1, \ldots, 50$ and comment on the results. [Computer] d) The previous parts show that using the quadratic equation to find the two roots gives poor results for large b; what is a practical alternative to accurately compute both roots? [Pencil]

1.5.X4 The Poisson distribution,

$$P[n;\alpha] = \frac{\alpha^n e^{-\alpha}}{n!}$$

gives the probability of n "events" (e.g., radioactive decays) given that the average number of events is α and each event occurs independently. Write a routine that computes and returns $P[n;\alpha]$. What is more likely, having no events occur when $\alpha = 10$ or having exactly 10^6 events occur when $\alpha = 10^6$? [Computer]

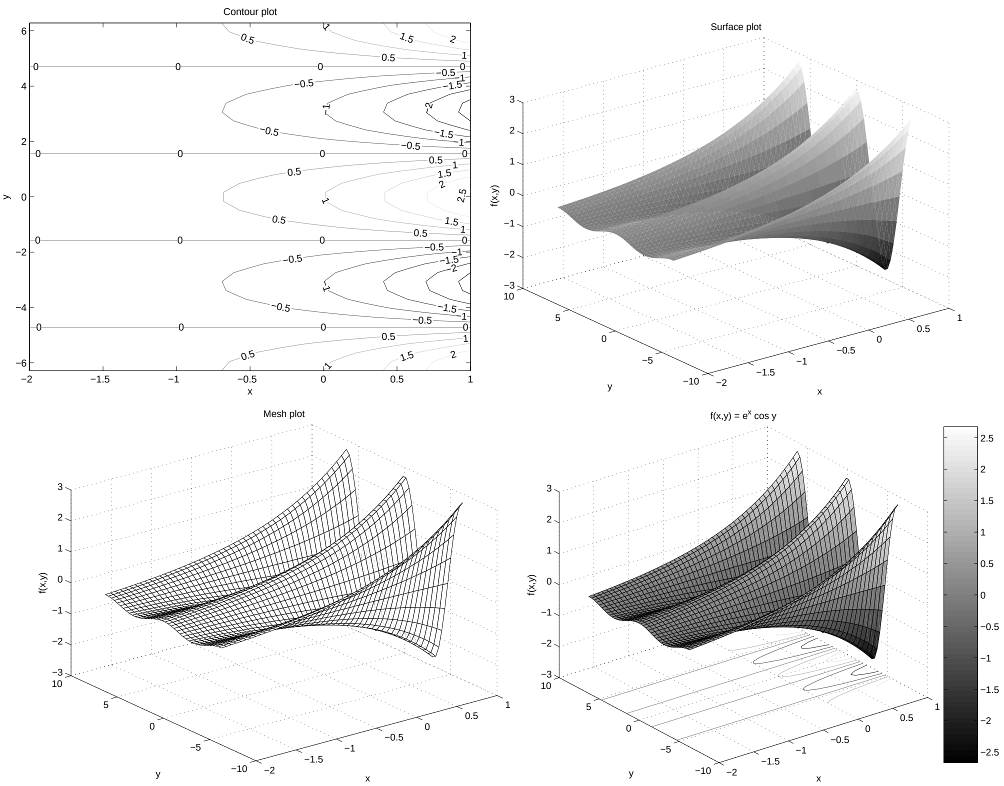

Figure 1:

Chapter 2

2.1.X1 Modify `balle` to compute and plot the kinetic, potential, and total energy of the ball. Graph these energies as a function of time; take initial height of 1 m, initial speed of 50 m/s, and angles varying from 30 to 60 degrees. Use a time step such that, in the absence of air resistance, the total energy is conserved to within about 1%. With air resistance, what is the fractional change in the total energy for this range of angles?

2.1.X2 Consider the equation of motion of a particle in the case of linear drag, specifically,

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{1}{\theta}\mathbf{v}$$

where θ is a constant, which we call the damping time. a) Solve this ODE analytically to obtain $\mathbf{r}(t)$ and $\mathbf{v(t)}$ given the initial conditions $\mathbf{r}(0)$ and $\mathbf{v(0)}$. [Pencil] b) Write a program that solves these equations of motion using the Euler method, plotting $|\mathbf{r}(t)|/(|\mathbf{v(0)}|\theta)$ as a function of t/θ. Compare the numerical results with the exact solution for the cases $\tau = \theta$, $\frac{1}{2}\theta$, $\frac{1}{4}\theta$, and $\frac{1}{8}\theta$. Take the initial conditions $\mathbf{r}(0) = 0$ and $|\mathbf{v(0)}| = 1$; compute the solution up to $t = 10\theta$. [Computer] c) Repeat the previous part using Euler-Cromer method.

2.2.X1 Modify the `pendul` program to compute and plot the total (normalized) energy,

$$\tilde{E} = \frac{E}{mgL} = \frac{1}{2}\frac{L}{g}\,\omega^2 - \cos\theta$$

a) Show that for small angles and time steps the error in the energy grows linearly in time for the Euler method. b) Estimating the velocity in the Verlet scheme as

$$\omega_{n+1} = \frac{\theta_{n+1} - \theta_n}{\tau}$$

show that the scheme approximately conserves energy for the scenarios shown in Figs. 2.7 and 2.8 in the textbook. c) Repeat the previous part but using the estimate

$$\omega_{n+1} = \frac{3\theta_{n+1} - 4\theta_n + \theta_{n-1}}{2\tau}$$

which is the three-point forward difference formula (see exercise 2.1). [Computer]

2.2.X2 The period of a simple pendulum may be written as

$$T = 2\pi\sqrt{\frac{L}{g}}\;\frac{1}{M\left(\sqrt{1-\sin^2\frac{1}{2}\theta_\mathrm{m}}\,\right)}$$

where $M(x)$ is the arithmetic-multiplicative mean, defined as

$$M(x) = \lim_{n\to\inf} a_n \qquad \text{where} \qquad a_1 = x,\;\; b_1 = 1,\;\; a_{n+1} = \frac{a_n + b_n}{2},\;\; b_{n+1} = \sqrt{a_n b_n}$$

a) Show that for small θ_m this formulation gives (2.38). [Pencil] b) Write a routine that computes $M(x)$ and use it in `pendul` to compute the theoretical period and compare it with the period estimated from the average time between reversals. [Computer]

Chapter 3

3.3.X1 Consider the following ODE,

$$\frac{dr}{dt} = \alpha r^2 - \beta r^3$$

a) If $r(t)$ is the radial size of an object at time t, give a physical interpretation for this ODE. [Pencil]
b) Find the change of variable that transforms the above ODE as, [Pencil]

$$\frac{dR}{dT} = R^2 - R^3$$

c) Write a program that uses `rk4` to compute $R(T)$. Plot the solution for the initial conditions $R(0) = 10^{-2}$, 10^{-3}, and 10^{-4} up to times 2×10^2, 2×10^3, and 2×10^4, respectively. In each case obtain the solution taking 1000 time steps. Comment on the results. [Computer] d) Repeat part c) using `rka`; also plot the time step τ used by the adaptive routine as a function of the iteration step. set the error tolerance to 10^{-4}. [Computer] e) Repeat the previous part but using MATLAB's stiff ODE solver, `ode23s`. [MATLAB]

Chapter 4

4.1.X1 Alan, Brian, and Ed go shopping. a) Alan buys a pizza, two beers, and three bags of chips, spending \$29. Brian buys three pizzas, three beers, and three bags, spending \$54; Ed buys four pizzas, five beers, and five bags, spending \$80. Using Gaussian elimination, find the price of each item. [Pencil or Computer] b) Show that if Ed buys six bags of chips instead of five that he spends \$83. [Pencil] c) Alan buys a pizza, two beers, and 3 bags of chips, spending \$29. Brian buys three pizzas, three beers, and three bags, spending \$54; Ed buys four pizzas, five beers, and six bags, spending \$83. By direct substitution, check that the price of pizza, beer, and bags of chips is \$9.50, \$6.00, and \$2.50. Reconcile this result with that of parts a) and b). [Pencil or Computer]

4.15+ Also compute the solution error, S, defined here as $S = \max\{|\Delta_i|\}$ where $\mathbf{\Delta} = \mathbf{a} - \mathbf{a}_c$, with $\mathbf{a}$ being the true solution and $\mathbf{a}_c = \mathbf{V}^{-1}\mathbf{y}$ being the computed solution. Do this for $y_i = 1$ for which we know the true solution is $a_1 = 1$, $a_i = 0$ for $i > 1$. Plot S versus N.

4.3.X1 a) Find a function, $f(x)$, which has a single root, x^*, for which Newton's method neither converges nor diverges. That is, for any initial guess the iterations of Newton's method oscillate equidistant about x^*. [Pencil] b) Write a program which plots this function and illustrates that Newton's method oscillates about the root. [Computer]

4.3.X2 The *secant method* of root finding is closely related to Newton's method. Specifically, given a pair of initial guesses, x_1 and x_2, the method iterates as,

$$x_{n+1} = x_n - \frac{f(x_n)}{d_n} \qquad \text{where} \qquad d_n = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

The advantage of the secant method is that only the function is required, not its derivative; the disadvantage is that the secant method usually converges more slowly. Write a program that implements the secant method and repeat exercise 4.16; for each case take $x_2 = x_1 + \frac{1}{10}$. Comment on the results. [Computer]

Chapter 5

5.2.X1 The SIDC-RWC Belgium World Data Center's website (`sidc.oma.be`) provides data on solar activity, such as sunspot number from 1749 to the present. a) Download the data set for monthly sunspot number (unsmoothed) and graph sunspot number versus time. b) Fit the data with a straight line and estimate the yearly rate of change of sunspot number. c) Remove the linear trend from the data then compute and plot the power spectrum; show that sunspot number varies cyclicly with a period of roughly 11 years. [Computer]

Chapter 6

Chapter 7

7.1.X1 Modify the `advect` program to use an initial condition of a square wave, initially centered at the origin, of width $L/2$,

$$f_0(x) = \begin{cases} 1 & |x| \leq L/4 \\ 0 & \text{otherwise} \end{cases}$$

Run your program for a variety of cases, including those in Figs. 7.3-7.7, and comment on the results.

Chapter 8

8.1.X1 In MATLAB the `relax` program takes significantly more CPU time to run when using Gauss-Seidel or SOR, as compared with Jacobi, even when the latter takes more iterations to converge. The difference is due to MATLAB's vectorization of some operations, such as `for` loops, when possible. Modify the Gauss-Seidel and SOR calculations in `relax`, following the form used for the Jacobi calculation, and demonstrate the resulting increase in computational speed. [MATLAB]

Chapter 9

Chapter 10

Chapter 11

11.2.X1 Monte Carlo integration selects random points instead of a regular grid, as in the quadrature methods considered in the previous chapter. Monte Carlo integration is particularly useful when the dimensionality of the integral is large or the boundaries are complicated; it is not particularly useful for simple integrals, such as

$$I = \int_a^b f(x)dx$$

which are evaluated as

$$I \approx I_{\mathrm{MC}} = \frac{b-a}{N}\sum_{i=1}^{N} f(x_i) \qquad \text{where} \qquad x_i = a + (b-a)\Re_i$$

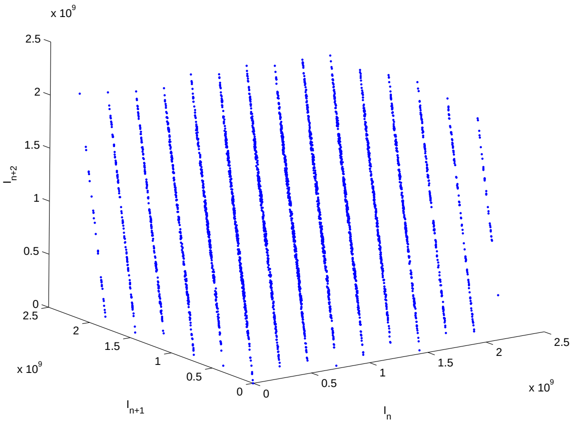

Figure 2:

Do exercise 10.13 using Monte Carlo integration, graphing the error as a function of N, the number of grid points. [Computer]

11.2.X2 Monte Carlo integration (see previous problem) is a simple way to estimate the area, A of an irregular surface. Specifically, define a surface by the function $f(x, y)$ such that $f(x, y) = 1$ if the point (x, y) is on the surface and zero if the point lies outside the surface. The Monte Carlo estimate of the area is

$$A \approx A_{\rm MC} = \frac{(x_{\rm max} - x_{\rm min})(y_{\rm max} - y_{\rm min})}{N} \sum_{i=1}^{N} f(x_i, y_i)$$

where $x_i = x_{\rm min} + (x_{\rm max} - x_{\rm min})\Re_i$, $y_i = y_{\rm min} + (y_{\rm max} - y_{\rm min})\Re'_i$. a)Explain why this works. [Pencil] b) Write a program that uses Monte Carlo integration to compute the area of the union of two unit circles whose centers are separated by a distance R. Verify your program for $R = 0$ and $R = 2$; estimate the area for $R = \frac{1}{2}$, 1, and $\frac{3}{2}$. [Computer]

11.3.X3 A linear congruential random number generator called `RANDU` was in common use in the 1960's. It used the coefficients $a = 65539 = 2^{16}+3$, $c = 0$, $M = 2^{31}$; these values were chosen because the multiplication could be done efficiently by a bit shift and an addition. Unfortunately, these coefficients yield a notoriously poor generator, which wasn't noticed for years. a) Show that the sequence generated by `RANDU` may be written as

$$I_{n+2} = (AI_{n+1} - BI_n) \bmod 2^{31}$$

and find the values of A and B, which are small ($A, B < 10$) positive integers. [Pencil] b) Write a program to generate the `RANDU` sequence and verify the result from the previous part. [Computer] c) Use the `RANDU` sequence to generate "random" points in 3-dimensional space. Plot the points $\mathbf{r}_i = [I_{i+2}, I_{i+1}, I_i]$, $i = 1, 4, 7, \ldots$ and show that they fall in planes (see Fig. 2. d) Repeat the previous part using the recommended values of $a = 7^5$, $c = 0$, and $M = 2^{31} - 1$.

11.18+ (e) Repeat parts (a-d) using Vegas-style rules, "On the first roll if you roll a 7 or 11, you win but if you roll a 2, 3, or 12 then you lose. Any other roll (4, 5, 6, 8, 9, or 10) establishes your mark. You continue throwing until you either roll your mark (and win) or roll a 7 (and lose)."
